Classification of small $(0,1)$ matrices

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Abstract

Denote by $\mathcal{A}_n$ the set of square $(0,1)$ matrices of order n. The set $\mathcal{A}_n$, $n \leq 8$, is partitioned into row/column permutation equivalence classes enabling derivation of various facts by simple counting. For example, the number of regular $(0,1)$ matrices of order 8 is 10160459763342013440. Let $\mathcal{D}_n$, $\mathcal{S}_n$ denote the set of absolute determinant values and Smith normal forms of matrices from $\mathcal{A}_n$. Denote by a_n the smallest integer not in $\mathcal{D}_n$. The sets $\mathcal{D}_9$ and $\mathcal{S}_9$ are obtained; especially, $a_9 = 103$. The lower bounds for a_n, $10 \leq n \leq 19$, (exceeding the known lower bound $a_n \geq 2f_{n-1}$, where f_n is nth Fibonacci number) are obtained. Row/permutation equivalence classes of $\mathcal{A}_n$ correspond to bipartite graphs with n black and n white vertices, and so the other applications of the classification are possible.

Key words: $(0,1)$ matrices, Smith normal form, permutation equivalence, determinant range, classification
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1 Introduction

Let $\mathcal{A}_n$ denote the set of square $(0,1)$ matrices of order n. Hadamard maximum determinant problem is: find the maximum determinant among the matrices in $\mathcal{A}_n$. In this paper we consider a slightly more general problem: determine the set $\mathcal{D}_n = \{|detA| \mid A \in \mathcal{A}_n\}$.

It is known [1] that determinants of $(0,1)$ matrices of order n are related to determinants of ± 1 matrices of order $n+1$. If A is a $(0,1)$-matrix of order n, let $B = \Psi(A)$ be a ± 1-matrix of order $n+1$ obtained from A by replacing

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its 0 by -1, bordering with a row -1's on the top, and a column of 1's on the right. Clearly, Ψ is a one-to-one correspondence. By adding row 1 of B to each of the other rows of B, we see that $\det B = 2^n \det A$.

By the Hadamard inequality $|\det B| \le \sqrt{(n+1)^{n+1}}$, and therefore for all $A \in \mathcal{A}_n$ $|\det A| \le 2^{-n}\sqrt{(n+1)^{n+1}}$. The equality is attained if B is an Hadamard matrix, i.e. if $BB^T = (n+1)I_{n+1}$, where T denotes transposition, and I_n is the unit matrix of order n; for $n > 2$ this implies $n = 4k-1$. For upper bounds for determinants of $A \in \mathcal{A}_n$ see for example [2].

Let d_n denote the largest element in $\mathcal{D}_n$, and let a_n be the smallest integer not in $\mathcal{D}_n$. Craigen [3] shows that the set $\mathcal{D}_n$ is the interval $\{1, 2, \ldots, d_n\}$ for $n \le 6$, but not for $n = 7$, because $a_8 = 41 < d_8 = 56$; he suggests that $a_9 = 103$.

Some interesting sequences, related to $(0,1)$ matrices are found in [4]: A003432 (the sequence d_n), A013588 (the sequence a_n), A051752 (c_n, the number of matrices in $\mathcal{A}_n$ with the determinant d_n) and A055165 (m_n, the number of regular matrices in $\mathcal{A}_n$). A few first members of these sequences are given in the following table. The values of a_9, c_8, c_9 and m_8 seem to be new.

	A003432	A013588	A051752	A055165
n	d_n	a_n	c_n	m_n
1	1	2	1	1
2	1	2	3	6
3	2	3	3	174
4	3	4	60	22560
5	5	6	3600	12514320
6	9	10	529200	28836612000
7	32	19	75600	270345669985440
8	56	41	*195955200	*10160459763342013440
9	144	*103	*13716864000	
10	320			
11	1458			
12	3645			
13	9477			

In this paper, which is a continuation of [5], the matrices in $\mathcal{A}_n$, $n \le 8$, are

partitioned into row/column permutation equivalence classes, enabling the classification by ADV, and more precisely — by SNF (see section 2). Let $\mathcal{S}_n$ denote the set of SNF's of matrices in $\mathcal{A}_n$. In section 3 the sets $\mathcal{D}_9$ and $\mathcal{S}_9$ are determined. In section 4 the lower bounds for a_n, $10 \le n \le 19$ are obtained; c_n, $n \le 9$, are obtained in section 5.

We introduce now some notation. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same dimension $m \times n$, we say that $A < B$ if A is *lexicographically less than* B, i.e. if for some pair of indices (i, j) the first $i - 1$ rows of A and B are equal, the first $j - 1$ elements in the ith row of A and B are equal, and $a_{ij} < b_{ij}$. For example,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} < \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The smallest matrix in a set $\mathcal{A}$ is the representative of $\mathcal{A}$.

Denote by $P_{i,j}$ the permutation matrix obtained from I_n by exchanging the ith and jth row.

The matrices $A, B \in \mathcal{A}_n$ are *equivalent* [7], $A \sim B$, if B is be obtained from A by a sequence of elementary row/column operations of the following types: exchange of two rows/columns, multiplication of a row/column by -1, and addition/subtraction of a row/column to/from another row/column. Let SNF(A) denote the SNF of A. It is known that $A \sim B$ is equivalent to SNF(A) = SNF(B) (in [7] this statement is proved for polynomial matrices).

The SNF diag$(d_1, d_2, \ldots, d_n)$ is written simply as a vector $(d_1, d_2, \ldots, d_n)$. If diagonal elements of SNF are repeated, we use the shortened exponential notation. For example, $(1^3, 2, 0)$ is short $(1, 1, 1, 2, 0)$. If $s \in \mathcal{S}_n$, then we also say that the SNF-class s is the set $\{A \in \mathcal{A} \mid \mathrm{SNF}(A) = s\}$.

Let J_n denote the square matrix of order n with all elements equal to one.

2 Classification of $(0, 1)$ matrices of order 8 or less

The set $\mathcal{D}_n$ could be obtained by computing determinants of all $A \in \mathcal{A}_n$. A better approach is to group matrices with the same determinant, and then to compute the determinant of only one matrix in each group. It is useful to classify $\mathcal{A}_n$ into subsets with constant absolute determinant value(ADV), or into even smaller subsets with constant SNF. We now review some such partitions of $\mathcal{A}_n$.

Let Π_r denote the group of row permutations of matrices from $\mathcal{A}_n$. Permutations from Π_r preserve ADV.

The representative of the matrix A orbit is obtained from A by sorting its rows into a nondecreasing sequence. Rows of A correspond to binary numbers less than $N = 2^n$. Therefore, the number of orbits of Π_r in $\mathcal{A}_n$ is equal to $\binom{N+n-1}{n-1}$, i.e. the number of nondecreasing sequences of length n from $\{0, 1, \ldots, N-1\}$. Let Π denote the group of row and column permutations; Π also preserves ADV. The group Π induces an equivalence relation π over $\mathcal{A}_n$. We say that matrices A and B are permutationally equivalent, $A \sim_\pi B$, if they are in the same orbit of Π. Let A_π denote the representative of the matrix A equivalence class (π-class; we say shorter that A_π is a π-representative of A).

Example 1 *The π-representative of*

$$\begin{bmatrix} 1\,0\,1 \\ 1\,1\,0 \\ 1\,0\,0 \end{bmatrix}$$

is the matrix

$$\begin{bmatrix} 0\,0\,1 \\ 0\,1\,1 \\ 1\,0\,1 \end{bmatrix},$$

the smallest of all 36 *permutationally equivalent matrices.*

Let $\mathcal{A}_n^\pi$ denote the set of π-representatives in $\mathcal{A}_n$. In [8] it is shown that the number of π-classes in $\mathcal{A}_n$ is given by:

$$|\mathcal{A}_n^\pi| = \sum_{i_1+2i_2+\ldots+ni_n=n} \; \sum_{j_1+2j_2+\ldots+nj_n=n} C(i)C(j)\exp_2 \sum_{r,s=1}^{n} i_r j_s 2^{(r,s)}, \qquad (1)$$

where the summation is over all vectors $i = (i_1, i_2, \ldots, i_n)$, $j = (j_1, j_2, \ldots, j_n)$, and

$$C(i) = n!/(1^{i_1} i_1! \ldots n^{i_n} i_n!)$$

is the number n-permutations with i_r cycles of length r, $r = 1, 2, \ldots, n$; (r, s) denotes GCD of integers r, s. The values $|\mathcal{A}_n^\pi|$ are listed in Table 1; they are easily computed for quite a large n using, for example, UBASIC [9]. It is seen that p_n is close to $2^{n^2}/(n!)^2$ for $n \leq 15$. An effective algorithm to generate the representative A_π of a given matrix A (section 2.3) simplifies the classification of matrices, because it enables to deal with the small subset $\mathcal{A}_n^\pi$ of $\mathcal{A}_n$.

Table 1
The number of permutationally nonequivalent matrices in $\mathcal{A}_n$, $n \le 15$.

n	$(2^{n^2}/n!^2)/\lvert\mathcal{A}_n^\pi\rvert$	$\lvert\mathcal{A}_n^\pi\rvert$
1	1.00000	2
2	0.57143	7
3	0.39506	36
4	0.35892	317
5	0.41433	5624
6	0.52685	251610
7	0.65875	33642660
8	0.77266	14685630688
9	0.85533	21467043671008
10	0.91045	105735224248507784
11	0.94565	1764356230257807614296
12	0.96754	100455994644460412263071692
13	0.98088	19674097197480928600253198363072
14	0.98886	13363679231028322645152300040033513414
15	0.99358	31735555932041230032311939400670284689732948

2.1 Matrix extension

In order to classify matrices in $\mathcal{A}_n$ by ADV values, one has to select carefully the order by which determinants are computed. It is natural to start from matrices of order $n-1$, and then to extend them by one row and one column of ones and zeros in each possible way. For an arbitrary $B \in \mathcal{A}_{n-1}$, let $\text{bord}(B)$ denote the subset of $\mathcal{A}_n$, containing matrices with the upper left minor equal to B. We say that the matrices in $\text{bord}(B)$ are *obtained by extending* B; if $A \in \text{bord}(B)$, then A is an *extension* of B.

The calculation of determinants of all matrices in $\text{bord}(B)$ is an easy task. If $A \in \text{bord}(B)$, then A is of the form

$$A = \begin{bmatrix} B & y \\ x & b \end{bmatrix}, \tag{2}$$

where $x = [x_1\ x_2\ \dots\ x_{n-1}]$ and $y = [y_1\ y_2\ \dots\ y_{n-1}]^T$. Then [1]

$$\det A = b \det B - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i y_j \det B_{ij}, \qquad (3)$$

where B_{ij} is the cofactor of B, corresponding to a_{ij}.

Obviously,

$$\mathcal{A}_n = \{A \mid (B, x, y, b) \in \mathcal{A}_{n-1} \times \{0,1\}^{n-1} \times \{0,1\}^{n-1} \times \{0,1\}\}.$$

If we precompute cofactors B_{ij}, then determinant of each matrix from $\mathrm{bord}(B)$ is computed by only one addition: for the fixed x, the column y might traverse the set of possible values via a Gray code (so that in the sequence of y's each two subsequent vectors differ in exactly one position).

Williamson [1] noted that it is enough to let B cross the set of π-representatives in $\mathcal{A}_{n-1}$. Let $\mathrm{bord}_\pi(B)$ denote the set of π-representatives of matrices in $\mathrm{bord}(B)$.

Lemma 2 *If* $B \sim_\pi B'$ *then* $\mathrm{bord}_\pi(B) = \mathrm{bord}_\pi(B')$.

PROOF. Let $A \in \mathrm{bord}_\pi(B)$. If the row/column permutations, transforming B into B', are applied to the first $n-1$ rows/columns of A, then the matrix with the upper left minor equal to B' is obtained. Therefore, the matrix permutationally equivalent to A is obtained by extending B', meaning that A is permutationally equivalent to a matrix from $\mathrm{bord}(B')$, i.e. $A \in \mathrm{bord}_\pi(B')$. Analogously, $\mathrm{bord}_\pi(B') \subseteq \mathrm{bord}_\pi(B)$, and so $\mathrm{bord}_\pi(B') = \mathrm{bord}_\pi(B)$. $\square$

Not only determinants, but also SNF's of matrices in $\mathrm{bord}(B)$ can be efficiently computed. The preprocessing step is to compute $D = \mathrm{SNF}(B) = \mathrm{diag}(d_1, d_2, \dots, d_n)$, and the matrices P, Q, such that $PBQ = D$, $|\det P| = |\det Q| = 1$. In order to determine $\mathrm{SNF}(A)$ for an arbitrary $A \in \mathrm{bord}(B)$ of the form (2), we use the identity

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B & y \\ x & b \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D & Py \\ xQ & b \end{bmatrix}. \qquad (4)$$

Denote $xQ = [a_1\ a_2\ \dots\ a_n]$, $Py = [c_1\ c_2\ \dots\ c_n]^T$. Suppose $d_1 = d_2 = \cdots = d_k = 1$, for some k, $1 \le k \le n$. Transforming the matrix from the righthand side of (4) by subtracting the row i multiplied by c_i from the row n, $1 \le i \le k$, and then subtracting the column i multiplied by c_i from the

column n, $1 \le i \le k$, we derive that A is equivalent to

$$\left[\begin{array}{ccc|ccc|c} 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ \hline 0 & \ldots & 0 & d_{k+1} & \ldots & 0 & c_{k+1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & d_n & c_n \\ \hline 0 & \ldots & 0 & a_{k+1} & \ldots & a_n & b - \sum_{i=1}^{k} a_i c_i \end{array}\right]. \tag{5}$$

Hence, SNF(A) determination is reduced to determination of SNF of a matrix of order $n-k$. The special cases when $k \ge n-1$ are extremely simple, and they are not rare at all, because the corresponding SNF-classes are among the largest ones (at least for $n \le 9$). More generally, one can reduce a_i, c_i modulo d_i, $1 \le i \le \operatorname{rank} B$.

2.2 Φ-extension

Following Williamson [1], the approach based on extending π-representatives only, can be further improved.

For an arbitrary $A \in \mathcal{A}_n$ let $A' = X_i A$ denote the matrix with the ith row equal to the ith row of A, and with the row $j \ne i$ equal to the coordinatewise modulo two sum of jth and ith row of A. Equivalently, $A' = RAS$, where R is the matrix obtained from I_n by subtracting ith row from the others, and then by multiplying ith row by -1; S is the matrix obtained from I_n by changing sign of columns corresponding to ones in the ith row of A. A third equivalent definition of X_i [1] can be stated as follows: in the ± 1 matrix $B = \Psi(A)$ of order $n+1$, the rows 1 and $(i+1)$ are exchanged, then the first row is "normalized" to all ones by changing signs of appropriate columns. By applying Ψ^{-1}, the matrix A' is obtained. Therefore, application of X_i to A corresponds to a special row permutation in $\Psi(A)$ (followed by scaling). It is natural to denote the identity transformation by X_0, $X_0 A = A$.

The transformation X_i also preserves ADV. The composition of arbitrary two transformations X_i, X_j is equivalent to only one:

$$X_i(X_j A) = \begin{cases} P_{i,j}(X_i A), & \text{if } 1 \le i, j \le n \text{ if } i \ne j \\ A, & \text{if } 1 \le i = j \le n \end{cases}.$$

Let Φ_r denote the set of $(n+1)!$ transforms of the form PX_i, $0 \le i \le n$, where P is an arbitrary permutation matrix.

Theorem 3 *The set Φ_r is a transformation group of $\mathcal{A}_n$.*

PROOF. We have

$$X_i PA = PX_{p_i}A,$$

where p_i is the index of the row of A, which is moved to the position i after the left multiplication by P. Let P_1 and P_2 be the two permutation matrices and let p_j be the position to which P_1 moves the row j after the left multiplication. Then

$$P_2X_jP_1X_i = P_2P_1X_{p_j}X_i = \begin{cases} P_2P_1, & p_j = i \\ P_2P_1P_{p_j,i}X_{p_j}, & p_j \ne i \end{cases}.$$

If $P_1 = P$ is an arbitrary permutation matrix, $1 \le i \le n$, $P_2 = P^{-1}$, and $p_j = i$, then

$$(PX_i)^{-1} = P^{-1}X_j. \qquad \square$$

Clearly, each orbit of Φ_r contains at most $n+1$ orbits of Π_r.

The corresponding transformation AX_j over the columns of A (coordinatewise addition modulo two of the column i to all other columns) is defined by $AX_j = (X_jA^T)^T$. Let Φ_c denote the group generated by column permutations and column transformations $(\cdot)X_i$.

Let Φ be the group generated by the elements of groups Φ_r and Φ_c; it also preserves ADV and its size is $(n+1)!^2$. Matrices A and A' are said to be ϕ-equivalent, $A \sim_\phi A'$, if they belong to the same orbit of Φ. Equivalently, $A \sim_\phi A'$ if and only if there exist row and column permutations P, Q, and row and column transformations X_i, X_j, such that $A = PX_iA'X_jQ$. For an arbitrary $A \in \mathcal{A}_n$ let A_ϕ denote the ϕ-representative of A; ϕ-class of A is the orbit of Φ containing A.

Let $\mathrm{bord}_\phi(B)$ denote the set of ϕ-representatives of matrices in $\mathrm{bord}(B)$. Williams [1] noted that Φ and Π have similar properties: in order to obtain the set $\mathcal{A}_n^\phi$ of all ϕ-representatives in $\mathcal{A}_n$, it is enough to extend ϕ-representatives in $\mathcal{A}_{n-1}$.

Lemma 4 *If $B \sim_\phi B'$, then* $\mathrm{bord}_\phi(B) = \mathrm{bord}_\phi(B')$.

PROOF. If B and B' are ϕ-equivalent, then there exist $g \in \Phi$, transforming B into B'. Suppose $A \in \mathrm{bord}_\phi(B)$. Then there exists a matrix $A' \in \mathrm{bord}(B)$, $A' \sim_\phi A$. By applying g to upper left minor of A, the matrix $A'' \sim_\phi A'$, $A'' \in$

bord(B') is obtained. Therefore, $A \sim_\phi A''$, and $A \in \text{bord}(B')$. Because A is a ϕ-representative, we obtain $A \in \text{bord}_\phi(B')$, implying $\text{bord}_\phi(B) \subseteq \text{bord}_\phi(B')$. Analogously, $\text{bord}_\phi(B') \subseteq \text{bord}_\phi(B)$, and hence $\text{bord}_\phi(B) = \text{bord}_\phi(B')$. $\square$

2.3 Effective determination of π-representatives

The classification of matrices in $\mathcal{A}_n$ by extending matrices from $\mathcal{A}_{n-1}^{\phi}$ must be accompanied by an effective procedure to determine A_π and A_ϕ for an arbitrary $A \in \mathcal{A}_n$.

The matrix A_π is the smallest among the family of at most $n!$ matrices obtained by sorting rows of all the column permutations of A. Search is performed more efficiently by a branch-and-bound algorithm. If we know the first i rows of A_π (i.e. the row and column permutations P, Q such that the first i rows of PAQ are minimal), then the next row of A_π is a smallest column permutation (only permutations preserving the first i rows of PAQ are considered) of some of the remaining rows of PAQ.

Algorithm 1 *Branch-and-bound algorithm to determine A_π given $A \in \mathcal{A}_n$.*

Input : $A \in \mathcal{A}_n$
Output : A_π; *the permutation matrices P, Q, such that $PAQ = A_\pi$;*
count – the number of pairs (P, Q), such that $PAQ = A_\pi$;
$P^{(0)} \leftarrow I_n$; $Q^{(0)} \leftarrow I_n$; $A_\pi \leftarrow J_n$;
$i \leftarrow 0$;
count $\leftarrow 0$;
Optimize(i);

{*Continuation of the search for A_π starting from the row i of $P^{(i-1)}AQ^{(i-1)}$,*}
{*i.e. when the first $i-1$ rows are already chosen and permuted*}
Optimize(i)
Generate the minimal set of boundaries $\Sigma^{(i)} = (s_0^{(i)} = 0, s_1^{(i)}, \ldots, s_{k_i}^{(i)} = n)$
between adjacent columns of $P^{(i-1)}AQ^{(i-1)}$, such that the $(i-1)$-prefixes
of columns from $s_{j-1}^{(i)} + 1$ to $s_j^{(i)}$ are mutually equal, $1 \le j \le k_i$;
for $j = i$ *to* n **do**
$v_{jl} \leftarrow \sum_{r=s_{l-1}+1}^{s_l} \left(P^{(i-1)}AQ^{(i-1)}\right)_{j,r}$, $1 \le l \le k_i$; {*the number of* 1*'s* }
{*in positions from $s_{j-1}^{(i)} + 1$ to $s_j^{(i)}$ in the jth row of $P^{(i-1)}AQ^{(i-1)}$*}

$L^{(i)} \leftarrow$ *the list of indices of largest vectors $v_j = (v_{j1}, v_{j2}, \ldots, v_{jk_i})$, $i \le j \le n$;*
for *all* $j \in L^{(i)}$ **do** { *the candidates for the ith row of A_π*}
$P^{(i)} \leftarrow P_{i,j}P^{(i-1)}$; { *exchange the rows i and j*}

compute $Q^{(i)}$ *from* $Q^{(i-1)}$*, so that all* 1*'s in the part of the row* i *from* $s^{(i)}_{l-1}+1$ *to* $s^{(l)}_{i}$ *are moved to the right end of the part,* $1 \le l \le k_i$*;*
$\{$*hence preserving the first* $i-1$ *rows of* $P^{(i)}AQ^{(i)}\}$
compare the ith row of $P^{(i)}AQ^{(i)}$ *to the ith row of* A_π*:*
if *the ith row of* $P^{(i)}AQ^{(i)}$ *is less* **then**
copy the first i *rows from* $P^{(i)}AQ^{(i)}$ *into* A_π*;*
fill with ones the rest of A_π*;*
if $i = n$ **then** $P \leftarrow P^{(i)}$*;* $Q \leftarrow Q^{(i)}$*; count* $\leftarrow 1$*;* **else** Optimize$(i+1)$*;*
else if *the ith row of* $P^{(i)}AQ^{(i)}$ *greater* **then**
continue; $\{$*bound step: try the next row index from* $L^{(i)}\}$
else
if $i = n$ **then** *count* $\leftarrow$ *count* $+1$*;* **else** Optimize$(i+1)$*;*

Example 5 *Algorithm 1, applied to the matrix from Example 1, gives the same* π*-representative as obtained by trivial algorithm, see Figure 1.*

The Algorithm 1 is not efficient for extremely symmetric matrices, such as I_n: in that case bound step does not ever occur, because all the remaining rows are always equally good. Hence, Algorithm 1 must be improved, in order to detect some symmetries, and to avoid some unnecessary repetitions. Suppose that there remain l rows not included in A_π, and that the column classes defined by $\Sigma^{(n-l-1)}$ are such, that all column classes in the remaining rows are *uniform* (they contain either all ones or all zeros), except for at most one column class, which in that case has l columns, with the row and column sums both equal to $l-1$ or 1. Then, because of the symmetry, it is enough to put in $L^{(n-l-1)}$ only one of the l remaining rows. After the incorporation of this simple heuristic, the algorithm much more efficiently deals with the matrices such as I_n, the complement of I_n, and the other highly symmetric matrices.

Using Algorithm 1, it is possible to determine A_ϕ for an arbitrary $A \in \mathcal{A}_n$: it is enough to find π-representatives of all $(n+1)^2$ matrices X_iAX_j, $0 \le i, j \le n$, and then to choose the smallest among them.

One of the outputs from Algorithm 1 is the number of the pairs of row/column permutations, transforming A into A_π. That number is used to determine the size of the π-class of A^T, as it will be demonstrated below.

Consider the problem of counting the matrices in the π-class of an arbitrary $A \in \mathcal{A}_n$. For an arbitrary $B \in \mathcal{A}_n$ let B_0 denote the matrix obtained from B by sorting its rows. If A has i_k groups of k equal rows, $1 \le k \le n$, then the number of matrices that could be obtained from A by row permutations is

$$a = n!/\prod_{k=1}^{n} i_k!$$

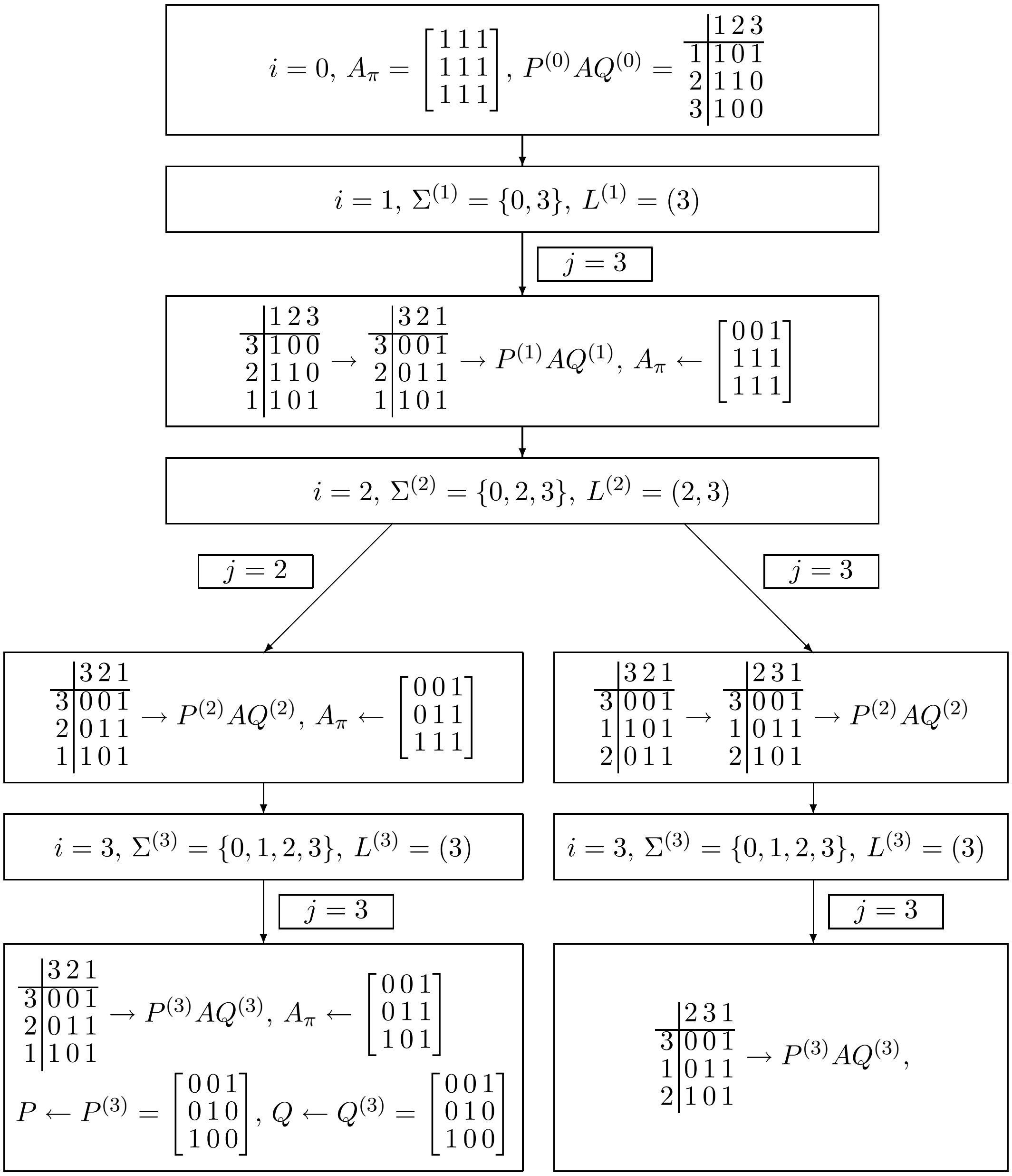

Fig. 1. An example of π-representative determination by Algorithm 1.

The representative of these a matrices is A_0. An arbitrary matrix A', obtained from A by a column permutation, generates in the same manner a new set of a matrices if and only if $A'_0 \neq A_0$. If the number of different matrices A'_0 is b, then the size of the π-class of A is ab. It is simpler to obtain b by counting the number p of column permutations A' of A satisfying $A'_0 = A_0$, because $b = n!/p$. Note that p is preserved by row and column permutations of A.

Applying Algorithm 1 to $(A^T)_\pi$, p is obtained even more easily. Indeed, suppose that A is already a π-representative, i.e. $A = A_\pi$. Then Algorithm 1 counts the row permutations A'' of A, such that there exists a column permutation A''' of A'', equal to A_π. Now we find $A' = ((A^T)_\pi)^T$ and apply Algorithm 1 (again)

to $(A')^T$. The matrix $(A')^T$ is a π-representative, because $((A')^T)_\pi = (A')^T$. Algorithm 1 gives the number of row permutations $(A'')^T$ of $(A')^T$, such that there exists a column permutation $(A''')^T$ of $(A'')^T$, equal to $(A')^T$. In other words, we obtain the number of column permutations A'' of A', such that there exists a row permutation A''' of A'', equal to A' — which is exactly p (*count* in Algorithm 1).

Example 6 *Looking again at Example 5, we see that there are two pairs* (P, Q) *that minimize* PAQ. *Therefore, there are* $3!^2/2 = 18$ *matrices in the* π*-class of* A^T.

The problem of counting the matrices in the SNF-class of an arbitrary $A \in \mathcal{A}_n$ is much harder. It is even harder is to enumerate the sets $\mathcal{A}_{n,k} = \{A \in \mathcal{A} \mid \operatorname{rank} A = k\}$, $0 \le k \le n$: (especially $m_n = \mathcal{A}_{n,n}$) We now explicitly enumerate the sets $\mathcal{A}_{n,1}$, $\mathcal{A}_{n,2}$, using the following characterization of matrices in $\mathcal{A}_{n,2}$.

Lemma 7 *If the matrix* $A \in \mathcal{A}_{n,2}$ *contains three different nonzero columns* a, b, c, *then one of them is equal to the sum of the other two, for example* $c = a + b$. *Furthermore, the set of nonzero rows of the matrix* $[a\ b]$ *equals to* $\{[0\ 1], [1\ 0]\}$. *There can not be four different nonzero columns in* A.

PROOF. Suppose $A \in \mathcal{A}_{n,2}$. If two nonzero columns of A are linearly dependent, then they are obviously equal. Suppose a, b, c are the three different nonzero linearly dependent columns, i.e. $\alpha a + \beta b + \gamma c = 0$ for some integers α, β, γ. The coefficients α, β, γ must be nonzero; otherwise, if for example $\alpha = 0$, then $\beta b + \gamma c = 0$ implies $b = c$. Denote by U the set of nonzero rows of the $n \times 3$ matrix $[a\ b\ c]$. Then

- $|U| > 1$; otherwise it would be $a = b = c$.
- $U \cap \{[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]\} = \emptyset$; if, for example $[1\ 0\ 0] \in U$, then $\alpha = 0$.
- therefore, $U \subseteq \{[1\ 1\ 1], [0\ 1\ 1], [1\ 0\ 1], [1\ 1\ 0]\}$ and $U \ne \{[1\ 1\ 1]\}$.
- $[1\ 1\ 1] \notin U$; if $[1\ 1\ 1] \in U$, and for example $[0\ 1\ 1] \in U$, then from $\alpha+\beta+\gamma = 0$ and $\beta + \gamma = 0$, it follows $\alpha = 0$.
- $U \ne \{[0\ 1\ 1], [1\ 0\ 1], [1\ 1\ 0]\}$; otherwise $\beta + \gamma = 0$, $\alpha + \gamma = 0$, $\alpha + \beta = 0$ implies $\alpha = \beta = \gamma = 0$.

Hence, there are three possibilities for U left: $\{[0\ 1\ 1], [1\ 0\ 1]\}$, or $\{[0\ 1\ 1], [1\ 1\ 0]\}$, or $\{[1\ 0\ 1], [1\ 1\ 0]\}$. If $U = \{[0\ 1\ 1], [1\ 0\ 1]\}$, then $\beta + \gamma = 0$, $\alpha + \gamma = 0$ implies $(\alpha, \beta, \gamma) = \gamma(-1, -1, 1)$, i.e. $c = a + b$; the set of nonzero rows of $[a\ b]$ is $\{[0\ 1], [1\ 0]\}$. The two other cases are symmetrical.

Suppose that A contains four different columns a, b, c, d. Then we must have for example $c = a + b$ and the set of nonzero rows of $[a\ b]$ is $\{[0\ 1], [1\ 0]\}$. Applying the first part of Lemma to a, b, d, we conclude that $d = a + b$ or $a = b + d$ or $b = a + d$. But $d = a - b$ and $d = b + a$ are impossible, and

$d = a + b$ implies $d = c$. The lemma is proved. $\square$

Theorem 8 *a) For an arbitrary $A \in \mathcal{A}_n$ the following three statements are equivalent:*

(1) $\operatorname{rank} A = 1$*;*
(2) $\operatorname{SNF}(A) = (1, 0^{n-1})$*;*
(3) A contains a column $a \neq 0$, such that all nonzero columns of A are equal to a.

The number of matrices in $\mathcal{A}_{n,1}$ equals

$$|\mathcal{A}_{n,1}| = (2^n - 1)^2.$$

b) For an arbitrary $A \in \mathcal{A}_n$ the following three statements are equivalent:

(1) $\operatorname{rank} A = 2$*;*
(2) $\operatorname{SNF}(A) = (1, 1, 0^{n-2})$*;*
(3) • *A contains the two nonzero columns $a \neq b$, such that all columns of A are in $\{0, a, b\}$, or*
 • *A contains the two nonzero columns $a \neq b$, such that the set of nonzero rows of $[a\ b]$ equals $\{[0\ 1], [1\ 0]\}$, and that the set of nonzero columns of A is $\{a, b, a + b\}$.*

The number of matrices in $\mathcal{A}_{n,2}$ equals

$$|\mathcal{A}_{n,2}| = (3^n - 2 \cdot 2^n + 1)(2 \cdot 4^n - 3 \cdot 3^n + 1)/2.$$

PROOF. a) If $\operatorname{rank} A = 1$ then A contains nonzero column a, such that all nonzero columns of A are equal to a. By subtracting one of nonzero columns from the others, we obtain an equivalent matrix with exactly one nonzero column a. By the column permutation column a is moved to the first position, and by the row permutation some 1 is moved to the upper left corner. By subtracting the first row from the other nonzero rows, we obtain that SNF of A is $(1, 0^{n-1})$. How many matrices of rank 1 there are? The number of choices for nonzero column a is $2^n - 1$, and the number of matrices corresponding to the fixed a is $2^n - 1$: each its column is 0 or a, but at least one of them has to be equal to a. Hence, $|\mathcal{A}_{n,1}| = (2^n - 1)^2$.

b) If $\operatorname{rank} A = 2$ then A contains two linearly independent columns, such that the other columns are their linear combinations. The number of different nonzero columns in A is either two or it is greater than two.

Case 1. Suppose there are exactly two different nonzero columns a, b in A.

The number of such matrices A is

$$\binom{2^n-1}{2}(3^n-2\cdot 2^n+1).$$

Indeed, the number of choices for a, b equals to the above binomial coefficient. Without loss of generality we suppose that $a<b$. For fixed a, b, by the inclusion-exclusion principle the number of matrices A is $3^n-2\cdot 2^n+1$, because

- 3^n is the number of matrices with the columns from the set $\{0,a,b\}$,
- 2^n is the number of matrices without a, and also the number of matrices without b,
- 1 is the number of matrices without a and b.

Case 2. If there are more than two different nonzero columns in A, then by Lemma 7 there are two different nonzero columns a, b ($a<b$) in A, such that the set of nonzero columns in A is $\{a,b,c=a+b\}$, and such that the row set of the matrix $[a\ b]$ is $\{[0\ 1],[1\ 0]\}$. There are $(3^n-2\cdot 2^n+1)/2$ choices for columns a, b satisfying these conditions. Indeed, consider all matrices $[a\ b]$, $[b\ a]$:

- 3^n is the number of matrices with the row set $\{[0\ 0],[0\ 1],[1\ 0]\}$,
- 2^n is the number of matrices without the row $[0\ 1]$, and also the number of matrices without the row $[1\ 0]$,
- 1 is the number of matrices without the rows $[0\ 1]$, $[1\ 0]$).

The number of matrices $[a\ b]$ is therefore $(3^n-2\cdot 2^n+1)/2$. The number $4^n-3\cdot 3^n+3\cdot 2^n-1$ of matrices with the set of nonzero columns $\{a,b,c\}$ (where $c=a+b$) is also obtained by the inclusion-exclusion principle:

- 4^n is the number of matrices with all the columns 0, a, b, c;
- 3^n is the number of matrices without the column a (and analogously without b, c);
- 2^n is the number of matrices without columns a, b (and analogously without a, c; and without b, c);
- 1 is the number of matrices without columns a, b, c.

Therefore, the number of matrices of the rank 2, with more than two different nonzero columns equals

$$(3^n-2\cdot 2^n+1)(4^n-3\cdot 3^n+3\cdot 2^n-1)/2.$$

The total number of matrices in $\mathcal{A}_{n,2}$ equals

$$\left(2\binom{2^n-1}{2}+(4^n-3\cdot 3^n+3\cdot 2^n-1)\right)(3^n-2\cdot 2^n+1)/2=$$
$$=(3^n-2\cdot 2^n+1)(2\cdot 4^n-3\cdot 3^n+1)/2.$$

In either case, in order to obtain $\mathrm{SNF}(A)$, the other nonzero columns are first transformed to 0 by subtracting a, b or $a+b$ from them. Next, in $[a\ b]$ there is a row $[0\ 1]$, because $a<b$; using that 1, the other elements of b are changed to

0. Finally, choosing some 1 in a, and subtracting if necessary that row from the others, after permuting rows/columns, we obtain the SNF. Hence, rank $A = 2$ implies $\mathrm{SNF}(A) = (1, 1, 0^{n-2})$. □

2.4 *Iterative classification of* $(0, 1)$ *matrices*

According to Lemma 2 we have

$$\mathcal{A}_{n+1}^{\pi} = \cup_{A \in \mathcal{A}_n^{\pi}} \mathrm{bord}_{\pi}(A).$$

By changing the order of calculations, it is possible to simplify repeated determination of π-representatives of matrices from $\mathrm{bord}(A)$ by Algorithm 1. Matrices B in $\mathrm{bord}(A)$ are of the form (2). For each y the π-representatives of B's corresponding to various inserted rows $[x\ b]$ are found spending smaller number of steps. The point is that the rows of the π-representative preceding the row $[x\ b]$ are already determined for some previous variants for that row.

Somewhat more detailed description follows. Determine first the π-representative of the matrix, corresponding to $x = 0$, $b = 0$; the inserted zero row $[x\ b]$ is certainly the first row in the π-representative. The corresponding row and column permutations P, Q are recorded. The remaining pairs (x, b) are then considered in turn, lexicographically ordered. The question arises, to which position l might $[x\ b]$ be moved during the π-representative determination, skipping the determination of first $l - 1$ rows of the representative. The obvious lower bound for l is the smallest among all positions where the previous rows w, obtained from $[x\ b]$ by changing exactly one 1 into 0, have been moved (except if there was an alternative to w during that step, i.e. if $L^{(i)}$ had more than one member at the moment when w arrived to its destination).

Instead of extending all $A \in \mathcal{A}_n^{\pi}$, it is enough to extend the matrices from the set $\mathcal{A}_n^{\phi}$ of all ϕ-representatives in $\mathcal{A}_n$. By extending all $A \in \mathcal{A}_n^{\phi}$ a subset of $\mathcal{A}_{n+1}^{\pi}$ is obtained; the set of ϕ-representatives of matrices from that subset is exactly $\mathcal{A}_{n+1}^{\phi}$.

It is convenient to use a balanced tree to collect π-representatives in an ordered fashion. We chose AVL tree [6] — the binary search tree satisfying the condition that, for every node, the difference between the heights of its left and right subtrees is at most 1. For $n = 8$, in order to save memory, a combination of AVL tree and the sorted array of matrices is used: from time to time the content of the tree is merged into the array. After collecting all π-representatives, the π-representatives set is reduced to the corresponding ϕ-representatives set. To determine the set of ϕ-representatives, corresponding to a given set of π-representatives, the following simple algorithm is used.

Algorithm 2 *Reduction of a given set L_π of π-representatives to the set L_ϕ of corresponding ϕ-representatives.*

{ *T — auxiliary AVL tree used to collect π-representatives.* }
while $L_\pi \neq \emptyset$
  while *there is a space in T for at least $(n+1)^2$ matrices*
    remove the first matrix A from L_π;
    generate the set T_A of π-representatives contained in the ϕ-class of A;
    insert T_A into T;
    insert A_ϕ into L_ϕ;
  remove from L_π all the matrices contained in T;
  $T \leftarrow \emptyset$;

The classification of $\mathcal{A}_8$ lasted about a month in parallel on five PC's. A huge number of collected π-representatives of order $n = 8$ caused serious difficulties. The space requirement is reduced by dividing π-representatives into subsets, according to their SNF. For each extended matrix, its SNF is determined, and the π-representatives are classified into subsets with the same SNF. These subsets are then independently processed. The hardest was the SNF-class $(1^7, 0)$, with 5204144555 π-representatives contained in a number of non disjoint subsets. These subsets were independently processed by Algorithm 2, producing the non disjoint sets of ϕ-representatives; their union consists of 71348129 ϕ-representatives, approximately 1/3 of matrices in $\mathcal{A}_8^\phi$.

In order to save the space, L_π and L_ϕ are stored in a sorted, compressed form: one byte for each matrix row; the group of consecutive matrices with the same first $n-2$ rows is stored so that the common $n-2$ rows are stored only once. As a result, the average space for a matrix of order 8 was little more than two bytes.

If somebody tries to extend ϕ-representatives of order 8, he could expect to process about 300 times more ϕ-representatives, each giving approximately 4 times more π-representatives. Therefore, the classification of matrices of order 9 is expected to last 1000 times longer, requiring huge memory.

2.5 Results of classification

We start with the simplest nontrivial case.

Example 9 *The 16 matrices of order 2 are divided into 3 ϕ-classes, which are further subdivided into 7 π-classes:*

$$\left\{\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}\right\},$$
$$\left\{\left\{\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}\right\},\left\{\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix}\right\},\right.$$
$$\left.\left\{\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix}\right\},\left\{\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}\right\},$$
$$\left\{\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\},\left\{\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix}\right\}\right\}.$$

In Table A.1 all the 36 π-representatives of order 3 are shown. The 5 SNF-classes are in separate blocks, divided into compartments with ϕ-classes. The first matrix in each ϕ-class is the smallest π-representative, i.e. the ϕ-representative. For each π-and SNF-class, their size is given. The matrices are represented by hexadecimal vectors, each component representing a row of a matrix. For example, the last vector $(3, 5, 6)$ in Table A.1 represents the matrix

$$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}.$$

The matrix $(1, 2, 5)$ is a π-representative of the matrix from Example 5.

In Table A.2 all the 39 ϕ-representatives of order 4 are shown, together with the sizes of their ϕ-classes.

In Table 2 ρ_n, $|\mathcal{A}_n^\phi|$, s_n, a_n, $|\mathcal{D}_n|$, and the set $\mathcal{D}_n$ are given for $1 \le n \le 8$, where $s_n = |\mathcal{S}_n|$ and $\rho_n = (2^{n^2}/(n+1)!^2)/|\mathcal{A}_n^\phi|$. In the last row of Table 2 s_9, $|\mathcal{D}_9|$, a_9, $\mathcal{D}_9$ are given; the explanation how they are obtained will be given in section 3.

Denote by $F(n)$ the following statement:

$$\begin{array}{l}A \in \mathcal{A}_n, \text{ satisfying } \mathrm{SNF}(A) = d = (d_1, d_2, \ldots, d_n) \text{ exists if and only if} \\ \text{there exists } A' \in \mathcal{A}_{n+1}, \text{ satisfying } \mathrm{SNF}(A') = d' = (d_1, d_2, \ldots, d_n, 0).\end{array} \quad (6)$$

Obviously, the first condition implies the second one. The implication in the opposite direction is not obvious at all; it would follow from the following stronger statement:

$H(n)$: Let $A' \in \mathcal{A}_{n+1}$, $\operatorname{rank} A' = n$, and $\mathrm{SNF}(A') = d' = (d_1, d_2, \ldots, d_n, 0)$. Then A' has at least one minor $A \in \mathcal{A}_n$ with $\mathrm{SNF}(A) = d = (d_1, d_2, \ldots, d_n)$.

Table 2
The numbers of equivalence classes in $\mathcal{A}_n$.

n	ρ_n	$\lvert\mathcal{A}_n^\phi\rvert$	s_n	a_n	$\lvert\mathcal{D}_n\rvert$	$\mathcal{D}_n$
1	0.250	2	2	2	2	$\{0,1\}$
2	0.148	3	3	2	2	$\{0,1\}$
3	0.074	12	5	3	3	$\{0-2\}$
4	0.117	39	8	4	4	$\{0-3\}$
5	0.167	388	14	6	6	$\{0-5\}$
6	0.334	8102	26	10	10	$\{0-9\}$
7	0.528	656103	56	19	22	$\{0-18,20,24,32\}$
8	0.701	199727714	129	41	46	$\{0-40,42,44,45,48,56\}$
9			333	103	114	$\{0-102,104,105,108,110,$ $112,116,117,120,125,128,144\}$

But the following matrix $F \in \mathcal{A}_{10}$ is a counterexample to $H(10)$:

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|cccccc} 0&0&1&1&1&0&1&0&0&1 \\ 0&1&1&0&0&1&0&1&0&1 \\ 1&1&0&0&0&1&1&0&1&0 \\ 1&0&0&1&1&0&0&1&1&0 \\ \hline 0&0&1&1&0&0&0&1&1&1 \\ 1&1&0&0&0&0&1&1&1&0 \\ 1&0&1&0&0&1&1&1&0&0 \\ 0&1&0&1&1&1&1&0&0&0 \\ 0&1&1&0&1&1&0&0&0&1 \\ 1&0&0&1&1&0&0&0&1&1 \end{array}\right].$$

The matrix F consists of blocks A, B, C, D, having $2, 3, 2, 3$ ones in each row respectively, and also having $2, 2, 3, 3$ ones in each column, respectively; F is singular, because the sums of rows of $[A\ B]$ and $[C\ D]$ are equal. It can be verified that $\operatorname{rank} F = 9$, $\mathrm{SNF}(F) = (1^9, 0)$, but all minors of F have SNF different from (1^9).

In Table A.3 the SNF-representatives of matrices in $\mathcal{A}_n$, $n \leq 8$, are listed, accompanied with the size measures of corresponding SNF-classes (the number of

Table 3
The number of matrices of the rank k in $\mathcal{A}_n$, $n \leq 8$.

n / k	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	1	9	49	225	961	3969	16129	65025
2		6	288	6750	118800	1807806	25316928	336954750
3			174	36000	3159750	190071000	9271660734	397046059200
4				22560	17760600	5295204600	1001080231200	144998212423680
5					12514320	34395777360	32307576315840	17952208799918400
6						28836612000	259286329895040	720988662376725120
7							270345669985440	7547198043595392000
8								10160459763342013440

matrices, the number of π-representatives and the number of ϕ-representatives in each SNF-class). The sizes of π-classes are determined using Algorithm 1. The classes are ordered lexicographically by the SNF (with zeros moved to the end of SNF).

One can verify this classification starting from the sorted list of all ϕ-representatives. For each of them one has to check if it is indeed a ϕ-representative. The next step is to sum the numbers of π-representatives in all ϕ-classes, and to compare the sum with the corresponding entry in Table 1. One could also check that the sum of sizes of SNF-classes in $\mathcal{A}_n$ equals 2^{n^2} for each $n \leq 8$, see Table A.3. The sorted lists of ϕ-representatives for $n \leq 8$ can be downloaded from http://www.matf.bg.ac.yu∼ezivkovm/01matrices.htm.

We now review some interesting facts, which are seen from Table A.3.

Let $T(n,k) = |\mathcal{A}_{n,k}|$. In Table 3 the numbers $T(n,k)$, $0 \leq k \leq n \leq 8$, are shown (of course, they are easily obtained from Table A.3). The part of Table 3 corresponding to $n \leq 7$ is the same as in [10]; it is also an entry in [4, Sequence A064230]. Another interesting entry in [4, Sequence A055165] is the sequence m_n, where m_n is the number of regular $(0,1)$ matrices of order n — the diagonal of Table 3. The seemingly new member of that sequence is $m_8 = 10160459763342013440$. If we suppose that all matrices in $\mathcal{A}_n$ are equiprobable, then the rank probability distribution is shown in Table 4 for $n \leq 8$. Looking at Table 4, one could erroneously conclude that large fraction of matrices in $\mathcal{A}_n$ is singular. In fact, the fraction of singular matrices in $\mathcal{A}_n$ tends to 0 for n large [11].

Table 4
The probability that a random matrix in $\mathcal{A}_n$ has the rank k, $0 \leq k \leq n \leq 8$.

n / k	1	2	3	4	5	6	7	8
0	0.5	0.0625	0.00195	0.00002	0.00000	0.00000	0.00000	0.00000
1	0.5	0.5625	0.09570	0.00343	0.00003	0.00000	0.00000	0.00000
2		0.3750	0.56250	0.10300	0.00354	0.00003	0.00000	0.00000
3			0.33984	0.54932	0.09417	0.00277	0.00002	0.00000
4				0.34424	0.52931	0.07706	0.00178	0.00001
5					0.37296	0.50052	0.05739	0.00097
6						0.41963	0.46059	0.03908
7							0.48023	0.40913
8								0.55080

Table 5
The possible numbers of π-orbits inside ϕ-orbits of $\mathcal{A}_n$.

n	The set of ϕ-orbit sizes
1	$\{1\}$
2	$\{1, 2, 4\}$
3	$\{1, 2, 4, 5, 9\}$
4	$\{1-5, 7, 9-11, 13, 16, 17\}$
5	$\{1-18, 20, 21, 25, 26, 30, 36\}$
6	$\{1, 2, 4-27, 29-32, 35-37, 42, 49\}$
7	$\{1-38, 40, 42-44, 48-50, 56, 64\}$
8	$\{1-46, 48-51, 53, 54, 56-58, 63-65, 72, 81\}$

It turns out that $F(n)$ (6) is true for $n \leq 7$, i.e. the set of SNF's of rank k is the same for all n, $k \leq n \leq 8$. For example, the SNF-representative of the SNF-class $(1, 1, 2, 0^{n-3})$ is the matrix $(0^{n-3}, 3, 5, 6)$ for $3 \leq n \leq 8$.

The smallest n for which there are two matrices in $\mathcal{A}_n$ with the same determinant, but with different SNF's is 5: SNF(3,C,15,16,19) $= (1, 1, 1, 4)$ and SNF(3,5,9,11,1E) $= (1, 1, 2, 2)$.

In Table 5 the possible numbers of π-orbits inside ϕ-orbits are shown for $1 \leq n \leq 8$. These numbers are between 1 and $(n+1)^2$; as it is seen, the value $(n+1)^2$ is attained only if $n \geq 5$.

Table 6
The maximal ADV's of matrices from $\mathcal{A}_{n+1}$, obtained by extending matrices equivalent to I_n.

n	$\vert\det A\vert$	A
3	3	3 5 9 E
4	5	3 5 E 16 19
5	9	3 D 15 1A 26 39
6	18	7 19 2A 34 4C 53 65
7	40	7 19 2A 56 65 9C B3 CB
8	105	7 39 5A AC D5 E3 136 14D 19B

If $A \in \mathcal{A}_n$, $A \sim I_n$ and $B \in \mathrm{bord}(A)$, then $\mathrm{SNF}(B)$ contains at least n ones, see (5). The question arises, what are the possible values of the last element of $\mathrm{SNF}(B)$, i.e. which values can take $|\det B|$? The largest possible values of $|\det B|$ under these assumptions, along with the examples of matrices B for which these values are attained, are given in Table 6. In fact, the matrices from Table 6 maximize $|\det B/\det A|$ for all regular $A \in \mathcal{A}$, $n \leq 8$.

More generally, it is interesting to describe the relationship of $\mathrm{SNF}(A)$ to $\mathrm{SNF}(A')$ if $A' \in \mathrm{bord}(A)$. During iterative classification, the sets

$$\{\mathrm{SNF}(B) \mid B \in \mathrm{bord}(A),\ A \in \mathcal{A}_n,\ \mathrm{SNF}(A) = s\}$$

are recorded for all SNF-classes $s \in \mathcal{S}_n$. The results are represented by the incidence matrix M_n of dimensions $|\mathcal{S}_n| \times |\mathcal{S}_{n+1}|$, with entries

$$m_{s,s'} = \begin{cases} 1, & \text{if there exist } A \in \mathcal{A}_n \text{ and } B \in \mathrm{bord}(A)\text{, with SNF's } s \text{ and } s' \\ 0, & \text{otherwise} \end{cases} . \tag{7}$$

Let $G(n)$, denote the following statement:

$G(n):$ There exist matrices $A \in \mathcal{A}_n$, $A' \in \mathrm{bord}(A)$, such that $\mathrm{SNF}(A) = (d_1, d_2, \ldots, d_n)$, $\mathrm{SNF}(A') = (d'_1, d'_2, \ldots, d'_n, d'_{n+1})$ if and only if there exist matrices $B \in \mathcal{A}_{n+1}$, $B' \in \mathrm{bord}(B)$, such that $\mathrm{SNF}(B) = (d_1, d_2, \ldots, d_n, 0)$, $\mathrm{SNF}(B') = (d'_1, d'_2, \ldots, d'_n, d'_{n+1}, 0)$. (8)

By exhaustive search it is verified that $G(n)$ is true for $n \leq 6$, enabling to put all the transposed incidence matrices M_n, $n \leq 7$ together into single Table A.4. The 1's are represented by $\bullet$; the 0's are represented by $\star$ if they are the

consequence of the following Lemma (describing constraints for $\mathrm{SNF}(A')$ if $A' \in \mathrm{bord}(A)$); otherwise, they are represented by $\circ$.

Lemma 10 *For an arbitrary $A \in \mathcal{A}_n$, let $A' \in \mathrm{bord}(A)$, and let $\mathrm{SNF}(A) = (d_1, d_2, \ldots, d_n)$, $\mathrm{SNF}(A') = (d'_1, d'_2, \ldots, d'_n, d'_{n+1})$. Then*

(1) $\operatorname{rank} A \leq \operatorname{rank} A' \leq \operatorname{rank} A + 2$*;*
(2) $d'_1 d'_2 \ldots d'_i$ *divides* $d_1 d_2 \ldots d_i$ *for all* i, $1 \leq i \leq \operatorname{rank} A$*;*
(3) $\prod_{i=1}^{n-1} d_i$ *divides* $\det A'$.

PROOF.

(1) The first inequality follows from the fact that the rank of a submatrix is a lower bound on the rank of a matrix. The second inequality follows from the observation that A' is an at most rank 2 perturbation of A.
(2) This is a direct consequence of the fact that $d'_1 d'_2 \ldots d'_i$ is the largest common divisor of all minors of A' of order i, see for example [7].
(3) Let P, Q be the matrices such that $\mathrm{SNF}(A) = PAQ = D = (d_1, d_2, \ldots, d_n)$, $|\det P| = |\det Q| = 1$. Let

$$A' = \begin{bmatrix} A & y \\ x & b \end{bmatrix}.$$

The case $\det A' = 0$ is trivial; suppose $\det A' \neq 0$. If $xQ = [a_1\ a_2\ \ldots a_n]$, $Py = [c_1\ c_2\ \ldots c_n]^T$, then from the identity

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & y \\ x & b \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D & Py \\ xQ & b \end{bmatrix}$$

it follows (another way to express determinants of matrices obtained by extension, see (3))

$$\det A' = b \prod_{i=1}^{n} d_i - \sum_{i=1}^{n} a_i c_i \prod_{\substack{1 \leq j \leq n, \\ j \neq i}} d_j. \tag{9}$$

Since $\operatorname{rank} A' = n+1$, then we have $\operatorname{rank} A \geq n-1$. If $\operatorname{rank} A = n-1$, then $d_n = 0$, implying

$$\det A' = -a_n c_n \prod_{i=1}^{n-1} d_i;$$

otherwise

$$\det A' = \left(b d_n - \sum_{i=1}^{n} a_i c_i d_n / d_i \right) \prod_{i=1}^{n-1} d_i.$$

In both cases $\prod_{i=1}^{n-1} d_i$ divides $\det A$. $\square$

Suppose $A \in \mathcal{A}_n$, $A' \in \operatorname{bord}(A)$. From Table A.4, we see the following interesting facts:

- The first $\circ$ in some M_n corresponds to $s = (1,0)$, $s' = (1,1,2)$. It is equivalent to following statement: if $A \in \mathcal{A}_{2,1}$ then $|\det A'| < 2$.
- if $A \in \mathcal{A}_{3,2}$, then $|\det A'| < 3$.
- if $A \in \mathcal{A}_4$, $\mathrm{SNF}(A) = (1,1,1,0)$, then $|\det A'| < 5$.
- if $A \in \mathcal{A}_4$, $\mathrm{SNF}(A) = (1,1,2,0)$, then $|\det A'| < 4$.
- if $A \in \mathcal{A}_5$, $\mathrm{SNF}(A) = (1,1,1,1,0)$, then $|\det A'| \neq 7$.
- if $A \in \mathcal{A}_5$, $\mathrm{SNF}(A) = (1,1,1,2,0)$, then $|\det A'| \neq 6$.
- if $A \in \mathcal{A}_5$, $\mathrm{SNF}(A) = (1,1,1,3,0)$, then $|\det A'| \notin \{6,9\}$.
- if $A \in \mathcal{A}_5$, $\mathrm{SNF}(A) = (1,1,1,2,2)$, then $\mathrm{SNF}(A') \neq (1,1,1,1,4,0)$.
- if $\mathrm{SNF}(A) = (1^{n-1}, d_n)$ and $\mathrm{SNF}(A') = (1^{n-1}, d'_n, 0)$ then d'_n divides d_n for all $n \leq 7$.
- if $\mathrm{SNF}(A) = s = (1^{n-1}, d_n) \in \mathcal{S}_n$ and $\mathrm{SNF}(A') = s' = (1^n, d'_{n+1}) \in \mathcal{S}_{n+1}$ then
  - · if $n \leq 6$, then $m_{s,s'} = 1$.
  - · if $n = 7$, then $m_{s,s'} = 1$ if and only if

$$(d_n, d'_{n+1}) \notin \{(17,34),(7,39),(13,39),(1,42),\\ (4,42),(6,42),(7,42),(13,42),(14,42)\}.$$

  - · if $n = 8$, then there are more exceptions to $m_{s,s'} = 1$, but there is one exotic group of them: if $d_n = 19$ then d'_{n+1} must be divisible by 19; 19 is the only integer satisfying such a condition.

3 Determinant and SNF sets of $(0,1)$ matrices of order 9

Determination of $\{|\det(A')| \mid A' \in \operatorname{bord}(A)\}$ is a simple operation, see the explanation following (3). It was effectively performed for all 199727714 matrices in $\mathcal{A}_8^{\phi}$; merging these sets $\mathcal{D}_9$ is obtained, see Table 2.

The similar idea — determine ADV's, and only if necessary, determine SNF's of the results of extension — is used to obtain $\mathcal{S}_9$. Suppose we know in advance the number f_d of different SNF's in $\mathcal{D}_9$ corresponding to a given ADV $d > 0$. During the extension of matrices from $\mathcal{A}_n$, the SNF's of extended matrices with the ADV d are determined only if the number of SNF's with ADV d is still less than f_d. If we know only upper bound on f_d, then the heuristic does not work — we have to determine SNF's of all matrices with the ADV d. Therefore, it is useful to determine f_d for at least some $d > 0$.

Table 7
The number of partitions of r into at most n positive integers.

n \ r	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4
3	1	1	2	3	4	5	7	8
4	1	1	2	3	5	6	9	11
5	1	1	2	3	5	7	10	13
6	1	1	2	3	5	7	11	14
7	1	1	2	3	5	7	11	15

Denote by $p_n(r)$ the number of partitions of r into at most n positive integers. In order to determine the upper bound for f_d, suppose first that d is a prime power, $d = p^r$. If $A \in \mathcal{A}_n$ and $|\det A| = d$, then SNF(A) is of the form

$$(p^{x_1}, p^{x_2}, \ldots, p^{x_n}), \quad 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n, \quad \sum_{i=1}^{n} x_i = r.$$

The number of different exponent vectors $(x_1, x_2, \ldots, x_n)$ is equal to $p_n(r)$. The values $p_n(m)$ are computed using the recurrence (see for example [12]) $p_n(0) = 1$, $n \geq 0$, $p_0(r) = 0$ for $r \geq 1$, and $p_n(r) = p_{n-1}(r) + p_n(r-n)$, see Table 7.

Example 11 *If $d = 8 = 2^3$ and $n = 6$ we have $p_6(3) = 3$;* SNF(A) *is one of* $(1^5, 8)$, $(1^4, 2, 4)$ *and* $(1^3, 2, 2, 2)$. *We see from Table A.3 that all these SNF do exist, i.e. for each of them there exists some* $(0,1)$ *matrix. Another example* $d = 3^2$, $n = 6$, *shows that* $p_6(2) = 2$ *is only an upper bound: the SNF-class* $(1^4, 3, 3)$ *is empty.*

More generally, if $d = \prod_{i=1}^{m} p_i^{\alpha_i}$, where p_i are different primes, then the upper bound on the number of different SNF's with the ADV d is $\prod_{i=1}^{m} p_n(\alpha_i)$.

Example 12 *If $n = 8$ and $d = 36$, then there are $p_8(2)p_8(2) = 4$ such SNF's:* $(1^6, 2, 18)$, $(1^7, 36)$, $(1^6, 3, 12)$, $(1^6, 6, 6)$*; all these SNF's are found in Table A.3.*

In order to obtain a tighter upper bound for the number of different SNF's, we have to include somewhat more information. If we further suppose that $A' \in \text{bord}(A)$ and SNF$(A) = s = (d_1, d_2, \ldots, d_n)$, then by Lemma 10 for some $s' = (d'_1, d'_2, \ldots, d'_{n+1})$ the equality SNF$(A') = s$ is impossible. For example, if

s contains k ones, then s' contains at least k ones and if rank $A \geq k+1$ then d'_{k+1} divides d_{k+1}.

Using these facts, the regular part of $\mathcal{S}_9$ was determined, see Table A.5. The ϕ-representatives from the chosen SNF-class of $\mathcal{A}_8$ were extended computing determinants, and, if necessary, determining SNF's. The upper bounds for the number of different SNF's, obtained by Lemma 10, are rough for larger ADV values, but the consequences are not dangerous, because of the small number of extended matrices with the large ADV: it is not hard to compute the SNF's of all of them.

To complete $\mathcal{S}_9$, it is necessary to determine the singular part of $\mathcal{S}_9$. If we would know that $F(8)$ is true, then the set of singular SNF's of order 8 would be equal to $\mathcal{S}_8$ (with each SNF extended by one zero, of course). Not knowing a simple proof of $F(8)$, we proceed with a shortened exhaustive proof.

The idea is to narrow the set of SNF-classes in $\mathcal{S}_8$, the extension of which can lead to a new singular SNF of order 9. If $\mathrm{SNF}(A) = (d_1, d_2, \ldots, d_8, 0)$ for some $A \in \mathcal{A}_9$, then (because we know the set of SNF's of lower orders) by Lemma 10 we can narrow the set SNF-classes, containing A. We obtain that the only new possible SNF's are the following SNF's of the rank 8: $(1^7, m, 0)$, $m = 44, 45, 48, 56$ and $(1^6, 2, 28, 0)$; and the following SNF's of the rank 7: $(1^6, 20, 0, 0)$, $(1^6, 24, 0, 0)$, $(1^6, 32, 0, 0)$, $(1^5, 2, 12, 0, 0)$, $(1^5, 2, 16, 0, 0)$, $(1^5, 4, 8, 0, 0)$, $(1^4, 2, 2, 8, 0, 0)$, $(1^4, 2, 4, 4, 0, 0)$.

The extension of which matrices gives the matrices with such SNF's? For example, we know that the SNF $(1^7, 44, 0)$ can be obtained only by the extension of a matrix in which 44 divides all minors of order 8; therefore 44 also divides a nonsingular minor of order 8; hence the SNF of that minor could be only $(1^6, 2, 22)$. Considering analogously the rest of listed SNF's of order 8, we obtain that matrices from $\mathcal{A}_{9,8}$, with the SNF equal to some from the list above, can be obtained only by the extension of matrices from $\mathcal{A}_8$ with the SNF $(1^6, 2, 22)$, $(1^6, 2, 24)$, $(1^6, 3, 15)$, $(1^5, 2, 2, 12)$, or $(1^5, 2, 2, 14)$.

Analogously, we obtain that matrices from $\mathcal{A}_{9,7}$ with one of the listed SNF's, can be obtained only by double extension of matrices from $\mathcal{A}_7$ with the SNF $(1^5, 2, 10)$, $(1^4, 2, 2, 6)$, or $(1^3, 2, 2, 2, 4)$. After the complete search through all matrices that can be obtained by the extensions listed, it is found that there are no new singular SNF's of order 9 i.e. that $F(8)$ is also true. That completes the determination of $\mathcal{S}_9$.

In Table A.6 the part of the incidence matrix M_8 is shown, corresponding to regular matrices in $\mathcal{S}_9$. The table was obtained by extending ϕ-representatives from $\mathcal{A}_{8,7}$ and $\mathcal{A}_{8,8}$; the singular extended matrices were ignored.

4 The lower bounds for the first missing determinant, a_n

Denote by f_n the nth Fibonacci number ($f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$). Paseman [13] shows that $a_n \geq 2f_{n-1}$. We give the sketch of his proof, and then we give the sharper lower bounds for a_n, $n \leq 19$.

Consider the so called Fibonacci matrices $F_n \in \mathcal{A}_n$ with the (i, j) element equal to 1 if and only if $j - i = -1, 0, 2, 4, \ldots$; $\det F_n = f_n$. The cofactors corresponding to the first row of F_n are f_{n-1}, f_{n-2}, $-f_{n-3}$, $-f_{n-4} \ldots, -f_1$. Consider the matrix $U \in \mathrm{bord}(F_n)$,

$$U = \begin{bmatrix} F_n & y \\ x & b \end{bmatrix},$$

where $x = [x_1\ x_2\ \ldots\ x_{n-1}]$, $y = [y_1\ y_2\ \ldots\ y_{n-1}]^T$. Let $y_1 = 1$, $y_2 = y_3 = \cdots = y_n = 0$ and $x_1 = x_2 = 0$. Then from (3) we have

$$\det U = \sum_{i=1}^{n-2} x_{n+1-i} f_i + b f_n.$$

Therefore, each integer from $[0, 2f_n - 1]$ is determinant of some $U \in \mathrm{bord}(F_n)$, and $a_n \geq 2f_{n-1}$.

In order to prove that $a_n \geq m$, one can give a list of matrices from $\mathcal{A}_{n-1}$, such that determinants of their extensions cover $[1, m-1]$. The proof verification then includes the procedure of finding determinants of all extensions of a given matrix. Still, such a list is essentially more compact than the list of matrices from $\mathcal{A}_n$, with determinants covering $[1, m-1]$.

Denote by a_A the minimal integer not in $\cup\{|\det B| \mid B \in \mathrm{bord}(A)\}$, the "extension spectrum" of $A \in \mathcal{A}_n$. In this context, the matrices A with high a_A are of special interest. If $a_A > 1$ and $\mathrm{SNF}(A) = (d_1, d_2, \ldots, d_n)$, then $d_1 = d_2 = \cdots = d_{n-1} = 1$, because determinants of all extensions of A are divisible by d_{n-1}, see (9).

In order to find lower bounds for some a_n, one can start from a well chosen set $\mathcal{B}_{n-1} \subset \mathcal{A}_{n-1}$, and then to find ADV's of all extended matrices. If m is the smallest number not equal to some of these ADV's, then $a_n \geq m$. Afterwards, some subset of extended matrices with different SNF's is taken to be the set $\mathcal{B}_n$, and the next iteration can be started.

The starting set $\mathcal{B}_9$ was constructed in the following way. From each SNF-class in $\mathcal{A}_8$ a number of matrices is taken, with different numbers of π representatives in their ϕ-classes. Extending these matrices, a set of matrices with

different SNF's is obtained, but without any matrix with the SNF $(1^8, 97)$. By adding one such matrix, the set $\mathcal{B}_9$ is completed. The sets $\mathcal{B}_{10}$, $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ are generated iteratively, as explained above. At the end, the ADV's of all matrices obtained by extending the matrices in $\mathcal{B}_{12}$ are determined. The resulting lower bounds are $a_{10} \geq 259$, $a_{11} \geq 739$, $a_{12} \geq 2107$, $a_{13} \geq 6157$.

For $n > 13$ we used an alternative heuristic, described by Algorithm 3.

Algorithm 3 *Heuristic to find lower bound for* a_{n+1}.

Input : $\mathcal{L}_n \subset \mathcal{A}_n$, *list of matrices to be extended.*
Output : *lower bound for* a_{n+1}*, and list* $\mathcal{L}_{n+1} \subset \mathcal{A}_{n+1}$
  of "promising" matrices for the following iteration.
{ *Initialization:* }
$first0 \leftarrow 1$; { *the first integer not "covered" by ADV's* }
$dmax \leftarrow 1$; { *the largest ADV found until now* }
$\mathcal{L}_{n+1} \leftarrow \emptyset$; { *output list* }
for *all* $A \in \mathcal{L}_n$ **do**
  { *Consider the extensions* $A' = \begin{bmatrix} A & y \\ & \\ x & b \end{bmatrix}$ }
  Compute $\det A$ *and* $B = [B_{ij}] = \operatorname{adj} A$; { *transposed cofactor matrix of* A}
  for *all* $y \in \{0,1\}^n$ **do**
    { *the next linear combination of rows of* B }
    determine the coefficients of the linear combination
      $-b \det A + \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_j B_{ij} \right)$
      and the sums s^+, s^- *of its positive and negative members;*
    if $\max\{-s^-, s^+\} \geq first0$ **then** {*"poor" linear combinations are skipped*}
      for *all* $(x, b) \in \{0,1\}^{n+1}$ **do**
        compute $\det A'$; { *by one addition only, using Gray code* }
        if $|\det A'| = first0$ **then**
          update $first0$;
        if $|\det A'| > dmax$ **then**
          $dmax \leftarrow |\det A'|$;
        if $|\det A'| > 0.9\, dmax$ **then**
          append A' *to* $\mathcal{L}_{n+1}$;

Elimination of "poor" linear combinations is a powerful heuristics if the matrices with the high extension spectra are placed in the beginning of $\mathcal{L}_n$. The major part of linear combinations is skipped after only a few first matrices in $\mathcal{L}_n$, reducing the extension complexity roughly to $O(n2^n)$ (instead of $O(4^n)$). In Table A.7 for $10 \leq n \leq 19$ we give

- lower bound for a_n,

- $|\mathcal{L}_{n-1}|$, the number of extended matrices,
- a matrix A_{n-1} with the highest extension spectrum found in $|\mathcal{A}_{n-1}|$,
- extension spectrum and determinant of A_{n-1}.

Complete lists of matrices, whose extension determinants prove these lower bounds, can be fount at http://www.matf.bg.ac.yu∼ezivkovm/01matrices.htm.

5 Counting $(0,1)$ matrices with the maximum determinant

Using the classification of $\mathcal{A}_n$, it is not hard to compute the number c_n [4, Sequences A051752] of matrices in $\mathcal{A}_n$ with the maximal determinant d_n (i.e. 1/2 of the number of matrices with the ADV d_n) for $n \leq 9$.

The first 8 members of the sequence c_n are found in Table A.3; the number $c_8 = 195955200$ is new.

In order to determine c_9, from Table A.4 we see that the matrix from $\mathcal{A}_9$ with the ADV 144 could be obtained only by extending matrices from $\mathcal{A}_8$ with the SNF $(1^5, 2, 2, 6)$ or $(1^5, 2, 2, 12)$. After the extension of these two SNF-classes, it turned out that there is a unique ϕ-class with the ADV 144 — the class with the representative (F,33,C3,FC,155,15A,166,196,1A9). Half of the number of matrices in that ϕ-class is $c_9 = 13716864000$. It is interesting that for all $n \leq 9$ there is a unique ϕ-class with the maximal ADV.

6 Acknowledgement

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A Large tables

Table A.1
π-representatives of $(0,1)$ matrices of order 3.

SNF			size
0	0	0	1
0	0	0	1

SNF			size
1	0	0	49
0	0	1	9
0	0	7	3
1	1	1	3
7	7	7	1
0	0	3	9
3	3	3	3
0	1	1	9
0	7	7	3
0	3	3	9

SNF			size
1	1	0	288
0	1	2	18
0	1	7	18
1	1	3	18
1	1	6	9
3	7	7	9
0	1	3	36
0	1	6	18
0	3	7	18
1	1	2	18
1	3	3	18
1	1	7	9
3	3	7	9
1	6	6	9
1	7	7	9
0	3	5	18
3	3	5	18
1	2	3	18
1	6	7	18

SNF			size
1	1	1	168
1	2	4	6
1	2	7	18
1	3	5	18
1	3	6	36
3	5	7	18
1	2	5	36
1	3	7	36

SNF			size
1	1	2	6
3	5	6	6

Table A.2

ϕ-representatives of $(0,1)$ matrices of order 4.

SNF				size
0	0	0	0	1
0	0	0	0	1

SNF				size
1	0	0	0	225
0	0	0	1	25
0	0	0	3	50
0	0	1	1	50
0	0	3	3	100

SNF				size
1	1	0	0	6750
0	0	1	2	200
0	0	1	3	400
0	0	1	6	600
0	0	3	5	600
0	0	3	7	300
0	1	1	2	600
0	1	1	6	450
0	1	2	3	600
0	1	3	3	300
0	1	6	6	900
0	1	6	7	900
0	3	3	5	900

SNF				size
1	1	1	0	35400
0	1	2	4	600
0	1	2	5	3600
0	1	2	7	1800
0	1	3	5	1800
0	1	3	6	3600
0	1	3	7	3600
0	1	6	A	3600
0	3	5	7	1800
0	3	5	9	1200
0	3	5	A	3600
1	2	3	4	3600
1	2	4	7	1200
1	2	5	6	3600
1	6	7	A	1800

SNF				size
1	1	2	0	600
3	5	6	0	600

SNF				size
1	1	1	1	20040
1	2	4	8	600
1	2	4	9	7200
1	2	5	A	1440
1	2	5	B	7200
1	2	7	B	3600

SNF				size
1	1	1	2	2400
1	6	A	C	2400

SNF				size
1	1	1	3	120
3	5	9	E	120

Table A.3
The representatives and the sizes of SNF-classes in $\mathcal{A}_n$, $n \leq 8$.

$\mathcal{A}_1$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
0	0	0	1	1	1	0
1	1	1	1	1	1	1
Total:			2	2	2	

$\mathcal{A}_2$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
0	0	0 0	1	1	1	0 0
1	0	0 1	9	4	1	0 1
2	1	1 1	6	2	1	1 2
Total:			16	7	3	

$\mathcal{A}_3$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
0	0	0 0 0	1	1	1	0 0 0
1	0	0 0 1	49	9	4	0 0 1
2	0	0 1 1	288	18	4	0 1 2
3	1	1 1 1	168	7	2	1 2 4
4	2	1 1 2	6	1	1	3 5 6
Total:			512	36	12	

$\mathcal{A}_4$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
0	0	0 0 0 0	1	1	1	0 0 0 0
1	0	0 0 0 1	225	16	4	0 0 0 1
2	0	0 0 1 1	6750	84	12	0 0 1 2
3	0	0 1 1 1	35400	150	14	0 1 2 4
4	0	0 1 1 2	600	5	1	0 3 5 6
5	1	1 1 1 1	20040	49	5	1 2 4 8
6	2	1 1 1 2	2400	10	1	1 6 A C
7	3	1 1 1 3	120	2	1	3 5 9 E
Total:			65536	317	39	

Table A.3
Continued

$\mathcal{A}_5$			The number of			The SNF-class representative				
	det	SNF	matrices	π-classes	ϕ-classes					
0	0	0 0 0 0 0	1	1	1	0	0	0	0	0
1	0	0 0 0 0 1	961	25	9	0	0	0	0	1
2	0	0 0 0 1 1	118800	260	37	0	0	0	1	2
3	0	0 0 1 1 1	3134400	1346	113	0	0	1	2	4
4	0	0 0 1 1 2	25350	25	5	0	0	3	5	6
5	0	0 1 1 1 1	16853400	2589	141	0	1	2	4	8
6	0	0 1 1 1 2	880200	210	17	0	1	6	A	C
7	0	0 1 1 1 3	27000	15	2	0	3	5	9	E
8	1	1 1 1 1 1	9702720	831	39	1	2	4	8	10
9	2	1 1 1 1 2	2427840	254	15	1	2	C	14	18
10	3	1 1 1 1 3	289440	51	5	1	6	A	12	1C
11	4	1 1 1 1 4	65520	12	2	3	5	9	11	1E
12	5	1 1 1 1 5	7200	3	1	3	5	E	16	19
13	4	1 1 1 2 2	21600	2	1	3	C	15	16	19
Total:			33554432	5624	388					

$\mathcal{A}_6$			The number of			The SNF-class representative					
	det	SNF	matrices	π-classes	ϕ-classes						
0	0	0 0 0 0 0 0	1	1	1	0	0	0	0	0	0
1	0	0 0 0 0 0 1	3969	36	9	0	0	0	0	0	1
2	0	0 0 0 0 1 1	1807806	660	76	0	0	0	0	1	2
3	0	0 0 0 1 1 1	189336000	7586	472	0	0	0	1	2	4
4	0	0 0 0 1 1 2	735000	86	10	0	0	0	3	5	6
5	0	0 0 1 1 1 1	5168108400	47605	1913	0	0	1	2	4	8
6	0	0 0 1 1 1 2	124744200	2120	115	0	0	1	6	A	C
7	0	0 0 1 1 1 3	2352000	91	9	0	0	3	5	9	E
8	0	0 1 1 1 1 1	30991962960	112080	3262	0	1	2	4	8	10
9	0	0 1 1 1 1 2	3122915040	14986	511	0	1	2	C	14	18
10	0	0 1 1 1 1 3	226603440	1618	75	0	1	6	A	12	1C
11	0	0 1 1 1 1 4	38419920	307	16	0	3	5	9	11	1E
12	0	0 1 1 1 1 5	3175200	46	3	0	3	5	E	16	19
13	0	0 1 1 1 2 2	12700800	78	4	0	3	C	15	16	19
14	1	1 1 1 1 1 1	18480102480	39637	952	1	2	4	8	10	20
15	2	1 1 1 1 1 2	7737327360	17642	442	1	2	4	18	28	30
16	3	1 1 1 1 1 3	1537446960	4079	128	1	2	C	14	24	38
17	4	1 1 1 1 1 4	628548480	1685	52	1	6	A	12	22	3C
18	5	1 1 1 1 1 5	127224720	429	18	1	6	A	1C	2C	32
19	6	1 1 1 1 1 6	93139200	263	9	3	5	9	16	2E	31
20	7	1 1 1 1 1 7	12877200	54	3	3	5	9	1E	2E	31
21	8	1 1 1 1 1 8	6703200	27	2	3	5	E	19	29	36
22	9	1 1 1 1 1 9	1058400	7	1	3	D	15	1A	26	39
23	4	1 1 1 1 2 2	208857600	473	17	1	6	18	2A	2C	32
24	8	1 1 1 1 2 4	3175200	12	1	3	C	15	1A	26	29
25	8	1 1 1 2 2 2	151200	2	1	7	19	1E	2A	2D	33
Total:			68719476736	251610	8102						

Table A.3
Continued

$\mathcal{A}_7$			The number of			The SNF-class representative						
	det	SNF	matrices	π-classes	ϕ-classes							
0	0	0 0 0 0 0 0 0	1	1	1	0	0	0	0	0	0	0
1	0	0 0 0 0 0 0 1	16129	49	16	0	0	0	0	0	0	1
2	0	0 0 0 0 0 1 1	25316928	1428	170	0	0	0	0	0	1	2
3	0	0 0 0 0 1 1 1	9254300328	31994	1908	0	0	0	0	1	2	4
4	0	0 0 0 0 1 1 2	17360406	246	34	0	0	0	0	3	5	6
5	0	0 0 0 1 1 1 1	989588124000	501563	17596	0	0	0	1	2	4	8
6	0	0 0 0 1 1 1 2	11359807200	13645	694	0	0	0	1	6	A	C
7	0	0 0 0 1 1 1 3	132300000	400	30	0	0	0	3	5	9	E
8	0	0 0 1 1 1 1 1	30826279895040	4358421	105808	0	0	1	2	4	8	10
9	0	0 0 1 1 1 1 2	1405763634240	316904	9295	0	0	1	2	C	14	18
10	0	0 0 1 1 1 1 3	64153434240	22902	853	0	0	1	6	A	12	1C
11	0	0 0 1 1 1 1 4	8175222720	3714	168	0	0	3	5	9	11	1E
12	0	0 0 1 1 1 1 5	509443200	413	23	0	0	3	5	E	16	19
13	0	0 0 1 1 1 2 2	2694686400	1032	58	0	0	3	C	15	16	19
14	0	0 1 1 1 1 1 1	219225571810560	13834240	261882	0	1	2	4	8	10	20
15	0	0 1 1 1 1 1 2	34159997168640	2624469	53874	0	1	2	4	18	28	30
16	0	0 1 1 1 1 1 3	4018162256640	376699	8633	0	1	2	C	14	24	38
17	0	0 1 1 1 1 1 4	1176364465920	123510	3024	0	1	6	A	12	22	3C
18	0	0 1 1 1 1 1 5	182858215680	23489	633	0	1	6	A	1C	2C	32
19	0	0 1 1 1 1 1 6	110954188800	13823	361	0	3	5	9	16	2E	31
20	0	0 1 1 1 1 1 7	12940704000	2133	64	0	3	5	9	1E	2E	31
21	0	0 1 1 1 1 1 8	5966553600	1006	33	0	3	5	E	19	29	36
22	0	0 1 1 1 1 1 9	829785600	189	7	0	3	D	15	1A	26	39
23	0	0 1 1 1 1 2 2	389700057600	37489	927	0	1	6	18	2A	2C	32
24	0	0 1 1 1 1 2 4	2857680000	415	19	0	3	C	15	1A	26	29
25	0	0 1 1 1 2 2 2	127008000	29	4	0	7	19	1E	2A	2D	33

Table A.3
Continued

$\mathcal{A}_7$			The number of			The SNF-class representative
	det	SNF	matrices	π-classes	ϕ-classes	
26	1	1 1 1 1 1 1 1	135491563468800	5593528	91764	1 2 4 8 10 20 40
27	2	1 1 1 1 1 1 2	83220427382400	3493129	58179	1 2 4 8 30 50 60
28	3	1 1 1 1 1 1 3	23436399974400	1020752	17707	1 2 4 18 28 48 70
29	4	1 1 1 1 1 1 4	13285672243200	581948	10189	1 2 C 14 24 44 78
30	5	1 1 1 1 1 1 5	3754520017920	172714	3169	1 2 C 14 38 58 64
31	6	1 1 1 1 1 1 6	4201407745920	185688	3320	1 6 A 12 2C 5C 62
32	7	1 1 1 1 1 1 7	813250851840	39068	749	1 6 A 12 3C 5C 62
33	8	1 1 1 1 1 1 8	693389168640	32490	645	1 6 A 1C 32 52 6C
34	9	1 1 1 1 1 1 9	257766405120	12609	253	1 6 1A 2A 34 4C 72
35	10	1 1 1 1 1 1 10	215881142400	10094	199	3 5 9 1E 2E 4E 71
36	11	1 1 1 1 1 1 11	49798425600	2598	55	3 5 9 1E 31 51 6E
37	12	1 1 1 1 1 1 12	67511808000	3263	71	3 5 E 16 39 59 66
38	13	1 1 1 1 1 1 13	12283084800	686	17	3 5 E 19 36 56 69
39	14	1 1 1 1 1 1 14	12260505600	615	12	3 5 19 29 36 4E 71
40	15	1 1 1 1 1 1 15	4064256000	215	6	3 D 15 26 38 5E 61
41	16	1 1 1 1 1 1 16	2235340800	143	6	3 C 15 36 39 5A 65
42	17	1 1 1 1 1 1 17	406425600	27	2	3 D 16 2E 39 5A 65
43	18	1 1 1 1 1 1 18	541900800	24	1	7 19 2A 34 4C 53 65
44	4	1 1 1 1 1 2 2	4413330432000	184475	3220	1 2 C 30 54 58 64
45	8	1 1 1 1 1 2 4	343226419200	15119	317	1 6 18 2A 34 4C 52
46	12	1 1 1 1 1 2 6	21946982400	997	28	3 5 19 29 36 4E 51
47	16	1 1 1 1 1 2 8	1219276800	102	6	3 C 31 55 5A 66 69
48	20	1 1 1 1 1 2 10	135475200	10	1	7 19 2A 34 4C 52 63
49	9	1 1 1 1 1 3 3	29940019200	1358	37	3 5 18 28 49 4E 71
50	18	1 1 1 1 1 3 6	139708800	10	2	3 1D 2D 36 3A 4E 71
51	16	1 1 1 1 1 4 4	254016000	19	3	3 C 35 3A 55 66 69
52	8	1 1 1 1 2 2 2	15969139200	750	29	1 E 32 3C 54 5A 66
53	16	1 1 1 1 2 2 4	118540800	20	4	3 C 30 55 5A 66 69
54	24	1 1 1 1 2 2 6	9676800	5	1	7 19 2A 34 4C 52 61
55	32	1 1 1 2 2 2 4	151200	1	1	F 33 3C 55 5A 66 69
Total:			562949953421312	33642660	656103	

Table A.3
Continued

$\mathcal{A}_8$			The number of			The SNF-class representative							
	det	SNF	matrices	π-classes	ϕ-classes								
0	0	0 0 0 0 0 0 0 0	1	1	1	0	0	0	0	0	0	0	0
1	0	0 0 0 0 0 0 0 1	65025	64	16	0	0	0	0	0	0	0	1
2	0	0 0 0 0 0 0 1 1	336954750	2800	295	0	0	0	0	0	0	1	2
3	0	0 0 0 0 0 1 1 1	396683821800	110064	5758	0	0	0	0	0	1	2	4
4	0	0 0 0 0 0 1 1 2	362237400	596	52	0	0	0	0	0	3	5	6
5	0	0 0 0 0 1 1 1 1	144191294561160	3696215	114651	0	0	0	0	1	2	4	8
6	0	0 0 0 0 1 1 1 2	801119894400	65292	2744	0	0	0	0	1	6	A	C
7	0	0 0 0 0 1 1 1 3	5797968120	1422	95	0	0	0	0	3	5	9	E
8	0	0 0 0 1 1 1 1 1	17559952974446400	88462953	1874266	0	0	0	1	2	4	8	10
9	0	0 0 0 1 1 1 1 2	379659804531840	3806686	96326	0	0	0	1	2	C	14	18
10	0	0 0 0 1 1 1 1 3	11124304309440	199374	6235	0	0	0	1	6	A	12	1C
11	0	0 0 0 1 1 1 1 4	1070841280320	27679	957	0	0	0	3	5	9	11	1E
12	0	0 0 0 1 1 1 1 5	50409475200	2430	108	0	0	0	3	5	E	16	19
13	0	0 0 0 1 1 1 2 2	350465875200	8104	265	0	0	0	3	C	15	16	19
14	0	0 0 1 1 1 1 1 1	669716034190338240	1150627540	18733404	0	0	1	2	4	8	10	20
15	0	0 0 1 1 1 1 1 2	46673270510307840	114940091	2053226	0	0	1	2	4	18	28	30
16	0	0 0 1 1 1 1 1 3	3467174719659840	11464276	226089	0	0	1	2	C	14	24	38
17	0	0 0 1 1 1 1 1 4	744008322193920	3048569	63679	0	0	1	6	A	12	22	3C
18	0	0 0 1 1 1 1 1 5	90294382946880	474025	10919	0	0	1	6	A	1C	2C	32
19	0	0 0 1 1 1 1 1 6	45671213145600	258700	5870	0	0	3	5	9	16	2E	31
20	0	0 0 1 1 1 1 1 7	4508891956800	33971	883	0	0	3	5	9	1E	2E	31
21	0	0 0 1 1 1 1 1 8	1853237232000	15033	409	0	0	3	5	E	19	29	36
22	0	0 0 1 1 1 1 1 9	225909129600	2474	74	0	0	3	D	15	1A	26	39
23	0	0 0 1 1 1 1 2 2	244678270233600	953097	19406	0	0	1	6	18	2A	2C	32
24	0	0 0 1 1 1 1 2 4	905427331200	6401	185	0	0	3	C	15	1A	26	29
25	0	0 0 1 1 1 2 2 2	37302249600	348	18	0	0	7	19	1E	2A	2D	33
26	0	0 1 1 1 1 1 1 1	5946448529329701120	5204144555	71348129	0	1	2	4	8	10	20	40
27	0	0 1 1 1 1 1 1 2	1255541169460515840	1268895126	18075782	0	1	2	4	8	30	50	60
28	0	0 1 1 1 1 1 1 3	201557577515938560	230051193	3411083	0	1	2	4	18	28	48	70
29	0	0 1 1 1 1 1 1 4	78772224791393280	98524334	1498752	0	1	2	C	14	24	44	78
30	0	0 1 1 1 1 1 1 5	16904181842714880	23305541	367457	0	1	2	C	14	38	58	64
31	0	0 1 1 1 1 1 1 6	15287255687531520	21926711	345484	0	1	6	A	12	2C	5C	62
32	0	0 1 1 1 1 1 1 7	2483072124099840	3947501	65221	0	1	6	A	12	3C	5C	62
33	0	0 1 1 1 1 1 1 8	1831786505418240	3014954	50240	0	1	6	A	1C	32	52	6C
34	0	0 1 1 1 1 1 1 9	601674419243520	1055591	17966	0	1	6	1A	2A	34	4C	72
35	0	0 1 1 1 1 1 1 10	451640549606400	806664	13584	0	3	5	9	1E	2E	4E	71
36	0	0 1 1 1 1 1 1 11	94403115878400	185595	3296	0	3	5	9	1E	31	51	6E
37	0	0 1 1 1 1 1 1 12	118893204864000	230334	4033	0	3	5	E	16	39	59	66
38	0	0 1 1 1 1 1 1 13	19770573312000	42957	793	0	3	5	E	19	36	56	69
39	0	0 1 1 1 1 1 1 14	18696085632000	39443	695	0	3	5	19	29	36	4E	71
40	0	0 1 1 1 1 1 1 15	5859844300800	12842	235	0	3	D	15	26	38	5E	61
41	0	0 1 1 1 1 1 1 16	3094524518400	7404	157	0	3	C	15	36	39	5A	65
42	0	0 1 1 1 1 1 1 17	526727577600	1380	27	0	3	D	16	2E	39	5A	65
43	0	0 1 1 1 1 1 1 18	702303436800	1376	24	0	7	19	2A	34	4C	53	65

Table A.3
Continued

$\mathcal{A}_8$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
44	0	0 1 1 1 1 1 2 2	25998918420787200	31764328	476566	0 1 2 C 30 54 58 64
45	0	0 1 1 1 1 1 2 4	904945412044800	1434616	23722	0 1 6 18 2A 34 4C 52
46	0	0 1 1 1 1 1 2 6	38868105830400	70882	1251	0 3 5 19 29 36 4E 51
47	0	0 1 1 1 1 1 2 8	1777705574400	4348	114	0 3 C 31 55 5A 66 69
48	0	0 1 1 1 1 1 2 10	175575859200	408	10	0 7 19 2A 34 4C 52 63
49	0	0 1 1 1 1 1 3 3	70619902617600	117219	1972	0 3 5 18 28 49 4E 71
50	0	0 1 1 1 1 1 3 6	181062604800	385	10	0 3 1D 2D 36 3A 4E 71
51	0	0 1 1 1 1 1 4 4	362125209600	902	23	0 3 C 35 3A 55 66 69
52	0	0 1 1 1 1 2 2 2	40931905996800	65455	1203	0 1 E 32 3C 54 5A 66
53	0	0 1 1 1 1 2 2 4	192036096000	606	26	0 3 C 30 55 5A 66 69
54	0	0 1 1 1 1 2 2 6	12541132800	66	5	0 7 19 2A 34 4C 52 61
55	0	0 1 1 1 2 2 2 4	195955200	5	1	0 F 33 3C 55 5A 66 69
56	1	1 1 1 1 1 1 1 1	3766962568171582080	2363927011	29610494	1 2 4 8 10 20 40 80
57	2	1 1 1 1 1 1 1 2	3107221856321587200	1958051993	24598561	1 2 4 8 10 60 A0 C0
58	3	1 1 1 1 1 1 1 3	1128344550375409920	717105693	9069162	1 2 4 8 30 50 90 E0
59	4	1 1 1 1 1 1 1 4	798113338051276800	508274669	6438161	1 2 4 18 28 48 88 F0
60	5	1 1 1 1 1 1 1 5	280558398045864960	180518667	2303868	1 2 4 18 28 70 B0 C8
61	6	1 1 1 1 1 1 1 6	391839981330309120	249971410	3172566	1 2 C 14 24 58 B8 C4
62	7	1 1 1 1 1 1 1 7	92717618729258880	60152437	772896	1 2 C 14 24 78 B8 C4
63	8	1 1 1 1 1 1 1 8	98081405804067840	63280071	811462	1 2 C 14 38 64 A4 D8
64	9	1 1 1 1 1 1 1 9	46392962843324160	30131275	388116	1 2 C 34 54 68 98 E4
65	10	1 1 1 1 1 1 1 10	49370839378882560	31886563	408631	1 6 A 12 3C 5C 9C E2
66	11	1 1 1 1 1 1 1 11	14214384381012480	9334363	121275	1 6 A 12 3C 62 A2 DC
67	12	1 1 1 1 1 1 1 12	25287474431600640	16375045	211035	1 6 A 1C 2C 72 B2 CC
68	13	1 1 1 1 1 1 1 13	6076097931697920	4012940	52419	1 6 A 1C 32 6C AC D2
69	14	1 1 1 1 1 1 1 14	8661618203857920	5648182	72918	1 6 A 32 52 6C 9C E2
70	15	1 1 1 1 1 1 1 15	4660876422921600	3060154	39958	1 6 1A 2A 4C 70 BC C2
71	16	1 1 1 1 1 1 1 16	3363977985177600	2214742	29108	1 6 18 2A 6C 72 B4 CA
72	17	1 1 1 1 1 1 1 17	1206477746611200	804732	10608	1 6 1A 2C 5C 72 B4 CA
73	18	1 1 1 1 1 1 1 18	2308713728025600	1503637	19409	1 E 32 54 68 98 A6 CA
74	19	1 1 1 1 1 1 1 19	548565282316800	367458	4865	3 5 9 31 51 6E 9E E1
75	20	1 1 1 1 1 1 1 20	883792178841600	583710	7689	3 5 E 19 36 68 A9 D6
76	21	1 1 1 1 1 1 1 21	420654153830400	280261	3719	3 5 E 19 36 69 A9 D6
77	22	1 1 1 1 1 1 1 22	358862424883200	237040	3085	3 5 E 32 56 69 99 E6
78	23	1 1 1 1 1 1 1 23	113325986995200	76671	1023	3 5 E 36 56 69 99 E6
79	24	1 1 1 1 1 1 1 24	228366581990400	152796	2117	3 5 18 68 A9 B6 CE D1
80	25	1 1 1 1 1 1 1 25	59747916211200	40146	532	3 5 18 2E 69 76 B1 CE
81	26	1 1 1 1 1 1 1 26	73204159795200	48393	630	3 5 19 2E 69 76 B1 CE
82	27	1 1 1 1 1 1 1 27	33155489203200	22046	293	3 5 39 59 6E 76 9E E1
83	28	1 1 1 1 1 1 1 28	34709152665600	23559	324	3 C 15 36 5A 65 B9 C6
84	29	1 1 1 1 1 1 1 29	9118971187200	6180	82	3 C 31 55 7A 96 D9 E5
85	30	1 1 1 1 1 1 1 30	28015323033600	18147	238	3 C 31 54 6A 9A A6 C9
86	31	1 1 1 1 1 1 1 31	3621252096000	2436	32	3 D 16 2E 5A 75 B9 C6

Table A.3
Continued

$\mathcal{A}_8$			The number of			The SNF-class
	det	SNF	matrices	π-classes	ϕ-classes	representative
87	32	1 1 1 1 1 1 1 32	5423648025600	3953	68	3 C 31 56 6A 9A A6 C5
88	33	1 1 1 1 1 1 1 33	2806470374400	1866	27	3 D 1E 35 66 79 AA D5
89	34	1 1 1 1 1 1 1 34	2831160729600	1815	23	3 D 31 54 6A 9A A6 C9
90	35	1 1 1 1 1 1 1 35	757170892800	491	7	3 1D 2E 56 69 9A B1 CD
91	36	1 1 1 1 1 1 1 36	1327792435200	873	13	3 D 35 56 69 99 AE C5
92	37	1 1 1 1 1 1 1 37	131681894400	81	1	7 19 2A 4C 71 A5 CB D6
93	38	1 1 1 1 1 1 1 38	592568524800	389	5	3 1D 2E 56 69 9A B5 CD
94	39	1 1 1 1 1 1 1 39	65840947200	45	1	7 19 2A 56 6D 9C B3 CB
95	40	1 1 1 1 1 1 1 40	263363788800	200	4	7 19 2A 4C 71 96 AD CB
96	42	1 1 1 1 1 1 1 42	65840947200	45	1	7 19 3E 63 AA B5 CC D2
97	4	1 1 1 1 1 1 2 2	264489939127895040	166494351	2095861	1 2 4 18 60 A8 B0 C8
98	8	1 1 1 1 1 1 2 4	48628694487582720	30927272	394107	1 2 C 30 54 68 98 A4
99	12	1 1 1 1 1 1 2 6	8317470133324800	5286249	67857	1 6 A 32 52 6C 9C A2
100	16	1 1 1 1 1 1 2 8	1671575454259200	1087782	14500	1 6 18 62 AA B4 CC D2
101	20	1 1 1 1 1 1 2 10	300128743756800	194974	2641	1 E 32 54 68 98 A4 C6
102	24	1 1 1 1 1 1 2 12	119718045619200	80651	1172	3 5 18 60 A9 B6 CE D1
103	28	1 1 1 1 1 1 2 14	14388990336000	9935	157	3 C 15 3A 65 A5 D6 D9
104	32	1 1 1 1 1 1 2 16	4180900147200	3259	65	3 C 31 56 6A 9A A6 C1
105	36	1 1 1 1 1 1 2 18	1360712908800	1103	23	3 D 31 54 6A 9A A6 C1
106	40	1 1 1 1 1 1 2 20	271593907200	232	5	3 D 35 59 6E 9E A9 C5
107	44	1 1 1 1 1 1 2 22	76814438400	74	2	3 1D 2E 56 79 9A B5 CD
108	48	1 1 1 1 1 1 2 24	16460236800	28	1	7 19 3E 61 AB B5 CC D2
109	9	1 1 1 1 1 1 3 3	5728974559056000	3662516	47056	1 6 A 30 50 92 9C E2
110	18	1 1 1 1 1 1 3 6	292630089830400	187420	2455	1 6 3A 5A 6C 74 9C E2
111	27	1 1 1 1 1 1 3 9	6466129689600	4234	71	3 C 31 54 7A 9A A6 C9
112	36	1 1 1 1 1 1 3 12	285310771200	219	7	3 1C 65 6A A6 B1 C9 D2
113	45	1 1 1 1 1 1 3 15	2743372800	6	1	7 39 5A 6C 9C AB B6 D1
114	16	1 1 1 1 1 1 4 4	282699080294400	186028	2556	1 6 18 6A 74 AA CC D2
115	32	1 1 1 1 1 1 4 8	724250419200	713	21	3 C 35 3A 56 69 99 A6
116	25	1 1 1 1 1 1 5 5	2339411155200	1497	22	3 D 15 26 5E 61 B8 C5
117	36	1 1 1 1 1 1 6 6	142655385600	136	4	7 19 2A 4B 74 8C D2 E1
118	8	1 1 1 1 1 2 2 2	2260349894476800	1421783	18397	1 2 1C 64 78 A8 B4 CC
119	16	1 1 1 1 1 2 2 4	136245037824000	90153	1346	1 6 18 60 AA B4 CC D2
120	24	1 1 1 1 1 2 2 6	6530011084800	5175	114	1 E 32 54 68 98 A4 C2
121	32	1 1 1 1 1 2 2 8	625488998400	578	15	3 D 31 55 6A 9A A6 C1
122	40	1 1 1 1 1 2 2 10	54867456000	96	5	3 D 31 56 6A 9A A6 C1
123	48	1 1 1 1 1 2 2 12	16460236800	49	3	3 1C 65 7A A9 B6 CE D1
124	56	1 1 1 1 1 2 2 14	391910400	6	1	3 1D 65 7A A9 B6 CE D1
125	32	1 1 1 1 1 2 4 4	98761420800	134	4	3 1C 64 78 A9 B2 CA D1
126	27	1 1 1 1 1 3 3 3	101504793600	96	6	7 19 2A 4B 74 8D B1 D6
127	16	1 1 1 1 2 2 2 2	1082717798400	754	17	3 1C 64 78 A9 AA B5 CD
128	32	1 1 1 1 2 2 2 4	28217548800	74	4	1 1E 66 78 AA B4 CC D2
Total:			18446744073709551616	14685630688	199727714	

Table A.4
Transposed incidence matrix M_7 containing all M_n, $1 \le n \le 7$. Symbol at position (s', s) carries information about $m_{s,s'}$ (7): •, ⋆, ∘ denotes respectively 1, 0 explained by Lemma 10, and 0 not explained by Lemma 10.

s s'	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 1	0 0 0 0 0 1 0 1	00 00 00 00 11 11 00 12	000 000 000 111 111 111 000 123	00000 00000 11111 11111 11111 11111 00000 12345	0 0 1 1 1 2 0 2	000000000 111111111 111111111 111111111 111111111 111111111 000000000 123456789	000 111 111 111 112 222 000 242	111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 000000000111111111 123456789012345678	11111 11111 11111 11111 11111 22222 00001 24680	1111111 1111111 1111111 1111112 1112222 3342222 0000000 3642464
0000000 0	●	·	·	··	···	·····	·	·········	···	··················	·····	·······
0000000 1	●	●	·	··	···	·····	·	·········	···	··················	·····	·······
0000001 1	●	●	●	··	···	·····	·	·········	···	··················	·····	·······
0000011 1	·	●	●	●●	···	·····	·	·········	···	··················	·····	·······
0000011 2	·	○	●	⋆●	···	·····	·	·········	···	··················	·····	·······
0000111 1	·	·	●	●●	●●●	·····	·	·········	···	··················	·····	·······
0000111 2	·	·	●	●●	⋆●⋆	·····	·	·········	···	··················	·····	·······
0000111 3	·	·	○	●●	⋆⋆●	·····	·	·········	···	··················	·····	·······
0001111 1	·	·	·	●⋆	●●●	●●●●●	●	·········	···	··················	·····	·······
0001111 2	·	·	·	●●	●●●	⋆●⋆●⋆	●	·········	···	··················	·····	·······
0001111 3	·	·	·	●⋆	●●●	⋆⋆●⋆⋆	⋆	·········	···	··················	·····	·······
0001111 4	·	·	·	●○	●●●	⋆⋆⋆●⋆	○	·········	···	··················	·····	·······
0001111 5	·	·	·	○⋆	●●●	⋆⋆⋆⋆●	⋆	·········	···	··················	·····	·······
0001112 2	·	·	·	●○	⋆●⋆	⋆⋆⋆⋆⋆	●	·········	···	··················	·····	·······
0011111 1	·	·	·	··	●⋆⋆	●●●●●	⋆	●●●●●●●●●	●●○	··················	·····	·······
0011111 2	·	·	·	··	●●⋆	●●●●●	●	⋆●⋆●⋆●⋆●⋆	●●●	··················	·····	·······
0011111 3	·	·	·	··	●⋆●	●●●●●	⋆	⋆⋆●⋆⋆●⋆⋆●	⋆⋆⋆	··················	·····	·······
0011111 4	·	·	·	··	●●⋆	●●●●●	●	⋆⋆⋆●⋆⋆⋆●⋆	○●○	··················	·····	·······
0011111 5	·	·	·	··	●⋆⋆	●●●●●	⋆	⋆⋆⋆⋆●⋆⋆⋆⋆	⋆⋆⋆	··················	·····	·······
0011111 6	·	·	·	··	●○○	●●●●●	●	⋆⋆⋆⋆⋆●⋆⋆⋆	⋆⋆⋆	··················	·····	·······
0011111 7	·	·	·	··	○⋆⋆	●●●●●	⋆	⋆⋆⋆⋆⋆⋆●⋆⋆	⋆⋆⋆	··················	·····	·······
0011111 8	·	·	·	··	●●⋆	●●●●●	●	⋆⋆⋆⋆⋆⋆⋆●⋆	⋆○○	··················	·····	·······
0011111 9	·	·	·	··	●⋆○	●●●●●	⋆	⋆⋆⋆⋆⋆⋆⋆⋆●	⋆⋆⋆	··················	·····	·······
0011112 2	·	·	·	··	●●⋆	⋆●⋆●⋆	●	⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●	··················	·····	·······
0011112 4	·	·	·	··	●●⋆	⋆●⋆●⋆	●	⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆●○	··················	·····	·······
0011122 2	·	·	·	··	⋆●⋆	⋆⋆⋆⋆⋆	●	⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆●	··················	·····	·······
0111111 1	·	·	·	··	···	●⋆⋆⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	●●●●●●●●●●●●●●●●●●	●●●●●	●●●○○○○
0111111 2	·	·	·	··	···	●●⋆⋆⋆	⋆	●●●●●●●●●	●●⋆	⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●	●●●●●	⋆●●●●●○
0111111 3	·	·	·	··	···	●⋆●⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆●⋆⋆●⋆⋆●⋆⋆●⋆⋆●⋆⋆●	⋆⋆●⋆⋆	●●⋆⋆⋆○⋆
0111111 4	·	·	·	··	···	●●⋆●⋆	○	●●●●●●●●●	●●○	⋆⋆⋆●⋆⋆⋆●⋆⋆⋆●⋆⋆⋆●⋆⋆	○●○●○	⋆⋆●○●○○
0111111 5	·	·	·	··	···	●⋆⋆⋆●	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆●⋆⋆⋆⋆●⋆⋆⋆⋆●⋆⋆⋆	⋆⋆⋆⋆●	⋆⋆⋆⋆⋆⋆⋆
0111111 6	·	·	·	··	···	●●●⋆⋆	⋆	●●●●●●●●●	●●⋆	⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆●	⋆⋆●⋆⋆	⋆●⋆⋆⋆●⋆
0111111 7	·	·	·	··	···	●⋆⋆⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 8	·	·	·	··	···	●●⋆●⋆	○	●●●●●●●●●	●●○	⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆●⋆⋆	⋆○⋆●⋆	⋆⋆○○○○○
0111111 9	·	·	·	··	···	●⋆●⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆⋆●	⋆⋆⋆⋆⋆	○○⋆⋆⋆⋆⋆
0111111 10	·	·	·	··	···	●○⋆⋆○	⋆	●●●●●●●●●	●●⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆●	⋆⋆⋆⋆⋆⋆⋆
0111111 11	·	·	·	··	···	○⋆⋆⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 12	·	·	·	··	···	●●●○⋆	○	●●●●●●●●●	●●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆	⋆⋆○⋆⋆	⋆⋆⋆⋆⋆○⋆
0111111 13	·	·	·	··	···	○⋆⋆⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 14	·	·	·	··	···	●○⋆⋆⋆	⋆	●●●●●●●●●	●●⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 15	·	·	·	··	···	●⋆○⋆○	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 16	·	·	·	··	···	●●⋆○⋆	○	●●●●●●●●●	●●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆	⋆⋆⋆○⋆	⋆⋆○⋆○⋆○
0111111 17	·	·	·	··	···	○⋆⋆⋆⋆	⋆	●●●●●●●●●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆
0111111 18	·	·	·	··	···	●●○⋆⋆	⋆	●●●●●●●●●	●●⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆●	⋆⋆⋆⋆⋆	⋆○⋆⋆⋆⋆⋆
0111112 2	·	·	·	··	···	●●⋆○⋆	●	⋆●⋆●⋆●⋆●⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆●●●●●
0111112 4	·	·	·	··	···	●●⋆●⋆	●	⋆●⋆●⋆●⋆●⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆●⋆●⋆	⋆⋆●○●○●
0111112 6	·	·	·	··	···	●●●○⋆	○	⋆●⋆●⋆●⋆●⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆●⋆⋆	⋆⋆⋆⋆⋆●⋆
0111112 8	·	·	·	··	···	●●⋆○⋆	○	⋆●⋆●⋆●⋆●⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆●⋆	⋆⋆○⋆○⋆○
0111112 10	·	·	·	··	···	○○⋆○○	○	⋆●⋆●⋆●⋆●⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆●	⋆⋆⋆⋆⋆⋆⋆
0111113 3	·	·	·	··	···	●⋆○⋆⋆	⋆	⋆⋆●⋆⋆●⋆⋆●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	●●⋆⋆⋆⋆⋆
0111113 6	·	·	·	··	···	●●○⋆⋆	⋆	⋆⋆●⋆⋆●⋆⋆●	⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆●⋆⋆⋆⋆⋆
0111114 4	·	·	·	··	···	●●⋆○⋆	○	⋆⋆⋆●⋆⋆⋆●⋆	○●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆●⋆○⋆○
0111122 2	·	·	·	··	···	⋆●⋆○⋆	●	⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆●●●●
0111122 4	·	·	·	··	···	⋆●⋆○⋆	○	⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆●⋆●
0111122 6	·	·	·	··	···	⋆○⋆○⋆	○	⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆●⋆
0111222 4	·	·	·	··	···	⋆⋆⋆⋆⋆	○	⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆●

Table A.4
Continued.

s	0 0 0 0 0 0	0 0 0 0 0 0	0 0 0 0 0 1	00 00 00 00 11 11	000 000 000 111 111 111	00000 00000 11111 11111 11111 11111	0 0 1 1 1 2	000000000 111111111 111111111 111111111 111111111 111111111	000 111 111 111 112 222	111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111	11111 11111 11111 11111 11111 22222	1111111 1111111 1111111 1111112 1112222 3342222
s'	0 0	0 1	0 1	00 12	000 123	00000 12345	0 2	000000000 123456789	000 242	000000000111111111 123456789012345678	00001 24680	0000000 3642464
1111111 1	·	·	·	· ·	· · ·	· · · · ·	·	●★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 2	·	·	·	· ·	· · ·	· · · · ·	·	●●★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 3	·	·	·	· ·	· · ·	· · · · ·	·	●★●★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 4	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★★★	○★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○★
1111111 5	·	·	·	· ·	· · ·	· · · · ·	·	●★★★●★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 6	·	·	·	· ·	· · ·	· · · · ·	·	●●●★★●★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	●●★★★★★
1111111 7	·	·	·	· ·	· · ·	· · · · ·	·	●★★★★★●★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 8	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★●★	○○○	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○○
1111111 9	·	·	·	· ·	· · ·	· · · · ·	·	●★●★★★★★●	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 10	·	·	·	· ·	· · ·	· · · · ·	·	●●★★●★★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 11	·	·	·	· ·	· · ·	· · · · ·	·	●★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 12	·	·	·	· ·	· · ·	· · · · ·	·	●●●●★●★★★	○★★	●●●●●●●●●●●●●●●●●●	●●●●●	●●●○○○★
1111111 13	·	·	·	· ·	· · ·	· · · · ·	·	●★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 14	·	·	·	· ·	· · ·	· · · · ·	·	●●★★★★●★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 15	·	·	·	· ·	· · ·	· · · · ·	·	●★●★●★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 16	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★●★	○○○	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○○
1111111 17	·	·	·	· ·	· · ·	· · · · ·	·	●★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 18	·	·	·	· ·	· · ·	· · · · ·	·	●●●★★●★★○	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	●●★★★★★
1111111 19	·	·	·	· ·	· · ·	· · · · ·	·	○★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 20	·	·	·	· ·	· · ·	· · · · ·	·	●●★○●★★★★	○★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○★
1111111 21	·	·	·	· ·	· · ·	· · · · ·	·	●★○★★★○★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 22	·	·	·	· ·	· · ·	· · · · ·	·	●○★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 23	·	·	·	· ·	· · ·	· · · · ·	·	○★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 24	·	·	·	· ·	· · ·	· · · · ·	·	●●●●★●★○★	○○○	●●●●●●●●●●●●●●●●●●	●●●●●	●●●○○○○
1111111 25	·	·	·	· ·	· · ·	· · · · ·	·	●★★★○★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 26	·	·	·	· ·	· · ·	· · · · ·	·	●○★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 27	·	·	·	· ·	· · ·	· · · · ·	·	●★●★★★★★○	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 28	·	·	·	· ·	· · ·	· · · · ·	·	●●★○★★○★★	○★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○★
1111111 29	·	·	·	· ·	· · ·	· · · · ·	·	○★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 30	·	·	·	· ·	· · ·	· · · · ·	·	●●●★●○★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	●●★★★★★
1111111 31	·	·	·	· ·	· · ·	· · · · ·	·	○★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 32	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★○★	○○○	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○○
1111111 33	·	·	·	· ·	· · ·	· · · · ·	·	●★○★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	●●★★★★★
1111111 34	·	·	·	· ·	· · ·	· · · · ·	·	○○★★★★★★★	★★★	●●●●●●●●●●●●●●●●○●	●●●●●	★★★★★★★
1111111 35	·	·	·	· ·	· · ·	· · · · ·	·	●★★★○★○★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 36	·	·	·	· ·	· · ·	· · · · ·	·	●●●●★○★★○	○★★	●●●●●●●●●●●●●●●●●●	●●●●●	●●●○○○★
1111111 37	·	·	·	· ·	· · ·	· · · · ·	·	○★★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	★★★★★	★★★★★★★
1111111 38	·	·	·	· ·	· · ·	· · · · ·	·	○○★★★★★★★	★★★	●●●●●●●●●●●●●●●●●●	●●●●●	★★★★★★★
1111111 39	·	·	·	· ·	· · ·	· · · · ·	·	○★○★★★★★★	★★★	●●●●●●○●●●●●○●●●●●	★★★★★	●○★★★★★
1111111 40	·	·	·	· ·	· · ·	· · · · ·	·	●●★○○★★○★	○○○	●●●●●●●●●●●●●●●●●●	●●●●●	★★●○○○○
1111111 42	·	·	·	· ·	· · ·	· · · · ·	·	○○○★★○○★★	★★★	○●●●○●○○●●●●○○●●●●	●●●●●	●○★★★★★
1111112 2	·	·	·	· ·	· · ·	· · · · ·	·	●●★○★★★★★	●★★	★●★●★●★●★●★●★●★●★●	●●●●●	★★○●●●★
1111112 4	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★○★	●●○	★●★●★●★●★●★●★●★●★●	●●●●●	★★●●●●○
1111112 6	·	·	·	· ·	· · ·	· · · · ·	·	●●●○★●★★★	●★★	★●★●★●★●★●★●★●★●★●	●●●●●	★●○●●●★
1111112 8	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★●★	●●○	★●★●★●★●★●★●★●★●★●	●●●●●	★★●●●●○
1111112 10	·	·	·	· ·	· · ·	· · · · ·	·	●●★○●★★★★	●★★	★●★●★●★●★●★●★●★●★●	●●●●●	★★○●●●★
1111112 12	·	·	·	· ·	· · ·	· · · · ·	·	●●●●★●★○★	●○○	★●★●★●★●★●★●★●★●★●	●●●●●	★●●●●●○
1111112 14	·	·	·	· ·	· · ·	· · · · ·	·	●●★○★★○★★	○★★	★●★●★●★●★●★●★●★●★●	●●●●●	★★○●●●★
1111112 16	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★○★	○●○	★●★●★●★●★●★●★●★●★●	●●●●●	★★●●●●○
1111112 18	·	·	·	· ·	· · ·	· · · · ·	·	●●○○★○★★○	●★★	★●★●★●★●★●★●★●★●★●	●●●●●	★●○●●●★
1111112 20	·	·	·	· ·	· · ·	· · · · ·	·	●●★○○★★○★	○○○	★●★●★●★●★●★●★●★●★●	●●●●●	★★●●●●○
1111112 22	·	·	·	· ·	· · ·	· · · · ·	·	○○★○★★★★★	○★★	★●★●★●★●★●★●★●★●★●	●●●●●	★★○●●●★
1111112 24	·	·	·	· ·	· · ·	· · · · ·	·	○●○○★○★○★	○○○	★○★○★○★●★●★○★●★●★○	●●●●○	★○○○●●○
1111113 3	·	·	·	· ·	· · ·	· · · · ·	·	●★●★★★★★○	★★★	★★●★★●★★●★★●★★●★★●	★★★★★	●●★★★★★
1111113 6	·	·	·	· ·	· · ·	· · · · ·	·	●●●★★●★★○	★★★	★★●★★●★★●★★●★★●★★●	★★●★★	●●★★★★★
1111113 9	·	·	·	· ·	· · ·	· · · · ·	·	●★●★★★★★○	★★★	★★●★★●★★●★★●★★●★★●	★★★★★	●●★★★★★
1111113 12	·	·	·	· ·	· · ·	· · · · ·	·	●●●●★○★★○	○★★	★★●★★●★★●★★●★★●★★●	★★●★★	●●★★★○★
1111113 15	·	·	·	· ·	· · ·	· · · · ·	·	○★○★○★★★○	★★★	★★●★★○★★○★★●★★○★★●	★★★★★	●●★★★★★
1111114 4	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★○★	●●○	★★★●★★★●★★★●★★★●★★	○●○●○	★★●○●○○
1111114 8	·	·	·	· ·	· · ·	· · · · ·	·	●●★●★★★○★	○●○	★★★●★★★●★★★●★★★●★★	○●○●○	★★●○●○○
1111115 5	·	·	·	· ·	· · ·	· · · · ·	·	●★★★○★★★★	★★★	★★★★●★★★★●★★★★●★★★	★★★★★	★★★★★★★
1111116 6	·	·	·	· ·	· · ·	· · · · ·	·	●●○○★○★★○	●★★	★★★★★●★★★★★●★★★★★●	★★●★★	★○★★★●★
1111122 2	·	·	·	· ·	· · ·	· · · · ·	·	★●★○★★★○★	●○●	★★★★★★★★★★★★★★★★★★	●●●●●	★★○●●●●
1111122 4	·	·	·	· ·	· · ·	· · · · ·	·	★●★●★★★○★	●●●	★★★★★★★★★★★★★★★★★★	●●●●●	★★●●●●●
1111122 6	·	·	·	· ·	· · ·	· · · · ·	·	★●★○★●★○★	●○●	★★★★★★★★★★★★★★★★★★	●●●●●	★★○●●●●
1111122 8	·	·	·	· ·	· · ·	· · · · ·	·	★●★●★★★○★	●●○	★★★★★★★★★★★★★★★★★★	●●●●●	★★●●●●●
1111122 10	·	·	·	· ·	· · ·	· · · · ·	·	★○★○★★★○★	○○○	★★★★★★★★★★★★★★★★★★	●●●●○	★★○●●●●
1111122 12	·	·	·	· ·	· · ·	· · · · ·	·	★●★○★○★○★	●○○	★★★★★★★★★★★★★★★★★★	●●●●●	★★○●●●●
1111122 14	·	·	·	· ·	· · ·	· · · · ·	·	★○★○★★★○★	○○○	★★★★★★★★★★★★★★★★★★	○○●●○	★★○●○●●
1111124 4	·	·	·	· ·	· · ·	· · · · ·	·	★●★●★★★○★	●●○	★★★★★★★★★★★★★★★★★★	★●★●★	★★●○●○●
1111133 3	·	·	·	· ·	· · ·	· · · · ·	·	★★●★★★★★○	★★★	★★★★★★★★★★★★★★★★★★	★★★★★	●●★★★★★
1111222 2	·	·	·	· ·	· · ·	· · · · ·	·	★★★★★★★★★	●○○	★★★★★★★★★★★★★★★★★★	★★★★★	★★★●●●○
1111222 4	·	·	·	· ·	· · ·	· · · · ·	·	★★★★★★★★★	●●●	★★★★★★★★★★★★★★★★★★	★★★★★	★★★●●●●

Table A.5
The part of $\mathcal{S}_9$ corresponding to nonsingular part of $\mathcal{A}_9$.

	det	SNF									
1-106	m	1	1	1	1	1	1	1	1	m	$m \in \{1-98, 100, 101, 102,\}$ $\cup\{104, 105, 108, 110, 120\}$
107-135	$4m$	1	1	1	1	1	1	1	2	$2m$	$m \in \{1-29\}$
136-148	$9m$	1	1	1	1	1	1	1	3	$3m$	$m \in \{1-13\}$
149-155	$16m$	1	1	1	1	1	1	1	4	$4m$	$m \in \{1-7\}$
156-159	25	1	1	1	1	1	1	1	5	$5m$	$m \in \{1-3, 5\}$
160	36	1	1	1	1	1	1	1	6	6	
161	72	1	1	1	1	1	1	1	6	12	
162	108	1	1	1	1	1	1	1	6	18	
163	49	1	1	1	1	1	1	1	7	7	
164	98	1	1	1	1	1	1	1	7	14	
165	64	1	1	1	1	1	1	1	8	8	
166	128	1	1	1	1	1	1	1	8	16	
167	81	1	1	1	1	1	1	1	9	9	
168	100	1	1	1	1	1	1	1	10	10	
169-182	$8m$	1	1	1	1	1	1	2	2	$2m$	$m \in \{1-13, 15\}$
183	32	1	1	1	1	1	1	2	4	4	
184	64	1	1	1	1	1	1	2	4	8	
185	96	1	1	1	1	1	1	2	4	12	
186	72	1	1	1	1	1	1	2	6	6	
187	128	1	1	1	1	1	1	2	8	8	
188	27	1	1	1	1	1	1	3	3	3	
189	54	1	1	1	1	1	1	3	3	6	
190	81	1	1	1	1	1	1	3	3	9	
191	108	1	1	1	1	1	1	3	3	12	
192	64	1	1	1	1	1	1	4	4	4	
193	128	1	1	1	1	1	1	4	4	8	
194	16	1	1	1	1	1	2	2	2	2	
195	32	1	1	1	1	1	2	2	2	4	
196	48	1	1	1	1	1	2	2	2	6	
197	64	1	1	1	1	1	2	2	2	8	
198	80	1	1	1	1	1	2	2	2	10	
199	96	1	1	1	1	1	2	2	2	12	
200	64	1	1	1	1	1	2	2	4	4	
201	144	1	1	1	1	1	2	2	6	6	
202	81	1	1	1	1	1	3	3	3	3	
203	32	1	1	1	1	2	2	2	2	2	
204	64	1	1	1	1	2	2	2	2	4	

Table A.6
The part of the transposed incidence matrix M_8, corresponding to the regular part of $\mathcal{S}_9$. Symbol at position (s', s) carries information about $m_{s,s'}$ (7): $\bullet$, $\star$, $\circ$ denotes respectively 1, 0 explained by Lemma 10, and 0 not explained by Lemma 10.

s' \ s	0111111 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 17 18	0111112 02 04 06 08	0111112 10, 0111113 03, 0111113 06, 0111114 04, 0111122 02, 0111122 04, 0111122 06, 0111222 04	1111111 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 42	1111112 02 04 06 08 10 12 14 16 18 20 22 24	1111113 03 06 09 12 15	1111114 04, 1111114 08, 1111115 05, 1111116 06	1111122 02 04 06 08 10 12 14	1111124 04, 1111133 03, 1111222 02, 1111222 04
1111111 1 1	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 2	●●★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 3	●★●★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 4	●●★●★★★★★★★★★★★★★★	○★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	★★★★
1111111 1 5	●★★★●★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 6	●●●★★●★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★★●	★★★★★★★	★★★★
1111111 1 7	●★★★★★●★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 8	●●★●★★★●★★★★★★★★★★	○○★★	★★★★○★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	○★○○
1111111 1 9	●★●★★★★★●★★★★★★★★★	★★★★	★○★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★○★★
1111111 1 10	●●★★●★★★★●★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 11	●★★★★★★★★★●★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 12	●●●●★●★★★★★●★★★★★★	○★○★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	●●★●	○○○○○○○	★★★★
1111111 1 13	●★★★★★★★★★★★●★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 14	●●★★★★●★★★★★★●★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 15	●★●★●★★★★★★★★★●★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★●★	★★★★★★★	★★★★
1111111 1 16	●●★●★★★●★★★★★★★●★★	○○★○	★★★○○○★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	○★○○
1111111 1 17	●★★★★★★★★★★★★★★★●★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 18	●●●★★●★★●★★★★★★★★●	★★★★	★○○★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★★●	★★★★★★★	★○★★
1111111 1 19	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 20	●●★●●★★★★●★★★★★★★★	○★★★	○★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●●★	○○○○○○○	★★★★
1111111 1 21	●★●★★★●★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 22	●●★★★★★★★★●★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 23	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 24	●●●●★●★●★★★●★★★★★★	○○○★	★★★★○★○★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	●●★●	○○○○○○○	○★○○
1111111 1 25	●★★★●★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 26	●●★★★★★★★★★★●★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 27	●★●★★★★★●★★★★★★★★★	★★★★	★○★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★○★★
1111111 1 28	●●★●★★●★★★★★★●★★★★	○★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	★★★★
1111111 1 29	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 30	●●●★●●★★★●★★★★●★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★●●	★★★★★★★	★★★★
1111111 1 31	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 32	●●★●★★★●★★★★★★★●★★	○○★○	★★★○○○★○	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	○★○○
1111111 1 33	●★●★★★★★★★●★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 34	●●★★★★★★★★★★★★★★○★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 35	●★★★●★●★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 36	●●●●★●★★●★★●★★★★★○	○★○★	★○○★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	●●★●	○○○○○○○	★○★★
1111111 1 37	●★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 38	●●★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 39	●★●★★★★★★★★★○★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 40	●●★●●★★●★●★★★★★★★★	○○★★	○★★★○★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●●★	○○○○○○○	○★○○
1111111 1 41	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 42	●●●★★●●★★★★★★○★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★★●	★★★★★★★	★★★★
1111111 1 43	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 44	●●★○★★★★★★●★★★★★★★	○★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	★★★★
1111111 1 45	●★●★●★★★○★★★★★○★★★	★★★★	★○★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★●★	★★★★★★★	★○★★
1111111 1 46	●○★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 47	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 48	●●●●★●★●★★★●★★★○★★	○○○○	★★★○○○○★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	●●★●	○○○○○○○	○★○○
1111111 1 49	●★★★★★○★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 50	●●★★●★★★★○★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 51	●★○★★★★★★★★★★★★★○★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 52	●●★○★★★★★★★★●★★★★★	○★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	★★★★
1111111 1 53	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 54	●●●★★●★★●★★★★★★★★○	★★★★	★○○★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★★●	★★★★★★★	★○★★
1111111 1 55	●★★★○★★★★★●★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 56	●●★●★★●○★★★★★●★★★★	○○★★	★★★★○★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	○★○○
1111111 1 57	●★○★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★★★★
1111111 1 58	●○★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 59	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 60	●●●●●●★★★●★○★★○★★★	○★○★	○★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	●●●●	○○○○○○○	★★★★
1111111 1 61	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 62	●○★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 63	●★●★★★●★○★★★★★★★★★	★★★★	★○★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★○★★
1111111 1 64	●●★●★★★●★★★★★★★○★★	○○★○	★★★○○○★○	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	○★○○
1111111 1 65	●★★★○★★★★★★★○★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 1 66	●●●★★○★★★★●★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	●●●●●	★★★●	★★★★★★★	★★★★
1111111 1 67	○★★★★★★★★★★★★★★★★★	★★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 1 68	●●★○★★★★★★★★★★★★○★	○★★★	★★★★★★★★	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	★★★★★	●●★★	○○○○○○○	★★★★

Table A.6
Continued.

s s'	000000000000000000 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 000000000111111111 123456789012345678	0000 1111 1111 1111 1111 1111 2222 0000 2468	00000000 11111111 11111111 11111111 11111112 11112222 23342222 10000000 03642464	11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 00000000011111111112222222222333333333344 12345678901234567890123456789012345678902	111111111111 111111111111 111111111111 111111111111 111111111111 111111111111 222222222222 000011111222 246802468024	11111 11111 11111 11111 11111 11111 33333 00011 36925	1111 1111 1111 1111 1111 1111 4456 0000 4856	1111111 1111111 1111111 1111111 1111111 2222222 2222222 0000111 2468024	1111 1111 1111 1111 1122 2322 4322 0000 4324
1111111 1 69	●⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 70	●●⋆⋆○⋆○⋆⋆○⋆⋆⋆○⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●○●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆●⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 71	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 72	●●●●⋆●⋆●○⋆⋆○⋆⋆⋆⋆⋆○	○○○⋆	⋆○○⋆○⋆○⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●●	●●⋆●	○○○○○○○	○○○○
1111111 1 73	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 74	○○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●○●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 75	●⋆●⋆●⋆⋆⋆⋆⋆⋆⋆⋆⋆○⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆●⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 76	●○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	○○○○○○○	⋆⋆⋆⋆
1111111 1 77	●⋆⋆⋆⋆⋆○⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 78	●●○⋆⋆○⋆⋆⋆⋆⋆⋆○⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●●	⋆⋆⋆●	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 79	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 80	●●⋆●●⋆⋆○⋆○⋆⋆⋆⋆⋆○⋆⋆	○○⋆○	○⋆⋆○○○⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●●⋆	○○○○○○○	○⋆○○
1111111 1 81	●⋆●⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆○⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆○⋆⋆
1111111 1 82	○○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 83	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 84	●●●●⋆○○⋆⋆⋆⋆○⋆○⋆⋆⋆⋆	○⋆○⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●●	●●⋆●	○○○○○○○	⋆⋆⋆⋆
1111111 1 85	●⋆⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆○⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆●⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 86	○○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 87	○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●○●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 88	●●⋆○⋆⋆⋆○⋆⋆●⋆⋆⋆⋆⋆⋆⋆	○○⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	○○○○○○○	○⋆○○
1111111 1 89	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 90	●●●⋆○●⋆⋆○○⋆⋆⋆⋆○⋆⋆○	⋆⋆⋆⋆	⋆○○⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●●	⋆⋆●●	⋆⋆⋆⋆⋆⋆⋆	⋆○⋆⋆
1111111 1 91	●⋆⋆⋆⋆⋆○⋆⋆⋆⋆⋆○⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 92	●○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	○○○○○○○	⋆⋆⋆⋆
1111111 1 93	○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●○●●●●●●●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 94	○○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●○●○●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 95	○⋆⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●●○○●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆●⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 96	●●○●⋆○⋆○⋆⋆⋆○⋆⋆⋆○⋆⋆	○○○○	⋆⋆⋆○○○○○	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●○	●●⋆●	○○○○○○○	○⋆○○
1111111 1 97	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●○○●●○●○●●○●●●○●●○○○●●●○●●●●○●○●○●○●○○○●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 98	●●⋆⋆⋆⋆○⋆⋆⋆⋆⋆⋆○⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●●●●●●●●●●●●●●●○●●●●●●●●●●●●●●●●○●●●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 100	●○⋆○○⋆⋆⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	○⋆⋆⋆⋆⋆⋆⋆	○●○●●●●●○●●●○●●●●●○●●●●●○●●●●●●●●●●●○●○●○	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●○⋆	○○○○○○○	⋆⋆⋆⋆
1111111 1 101	○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●○●●○○○●●●●●●●○○●●○○●●●●○○●●●●○●●○○○○●●●○	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 102	●○○⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆○⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●●●●○●○●●●●●●●●●●●○●●●●●●●●●●●●●●●●●●●●●●	●●●●●●●●●●●●	●●●●○	⋆⋆⋆●	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 104	○○⋆○⋆⋆⋆○⋆⋆⋆⋆○⋆⋆⋆⋆⋆	○○⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	●●○●○○●●○○●●○●●●○●○●●●○●○○●●○●●●●●○●●○○●○	○●●●●●●●○●○○	⋆⋆⋆⋆⋆	●●⋆⋆	○○○○○○○	○⋆○○
1111111 1 105	●⋆○⋆○⋆○⋆⋆⋆⋆⋆⋆⋆○⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	●○●●●●●○●○●●●●●●●●○●●●●●○●●●○●●●●●●●○●●●●	⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●●○○	⋆⋆○⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 108	○●●○⋆○⋆⋆○⋆⋆○⋆⋆⋆⋆⋆○	○⋆○⋆	⋆○○⋆⋆⋆⋆⋆	○●○●○○○●○●○●○●○●○●○●○●●●●●○○●○●●○○●●○○○●○	○●○○●●●○●●●●	○○○○○	○○⋆○	○○○○○○○	⋆○⋆⋆
1111111 1 110	○○⋆⋆○⋆⋆⋆⋆○●⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	○●○○●●○●○●○●●●●●○●○●○○●●○●●●○●●●○●●●●●○●○	○●○○●●●●●●○●	⋆⋆⋆⋆⋆	⋆⋆○⋆	⋆⋆⋆⋆⋆⋆⋆	⋆⋆⋆⋆
1111111 1 120	○○○○○○⋆○⋆○⋆○⋆⋆○⋆⋆⋆	○○○⋆	○⋆⋆⋆○⋆○⋆	○○●○●○○●○○○●○○●○○○○●○○○●○○○○○●○○○○●●○○○●●	○○●○○●○●○○○○	○●●○○	○●○○	○○○○○○○	○⋆○○
1111111 2 2	●●⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 4	●●⋆●⋆⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	●●⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 6	●●●○⋆●⋆⋆⋆⋆⋆○⋆⋆⋆⋆⋆⋆	●⋆●⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	○○⋆●	●●●●●●●	⋆⋆⋆⋆
1111111 2 8	●●⋆●⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆○⋆⋆	●●⋆●	⋆⋆⋆○○○⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 10	●●⋆○●⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆⋆	●⋆⋆⋆	●⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 12	●●●●⋆●⋆○⋆⋆⋆●⋆⋆⋆⋆⋆⋆	●●●⋆	⋆⋆⋆⋆○⋆○⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	●●⋆●	●●●●●●●	●⋆○○
1111111 2 14	●●⋆○⋆⋆●⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆	●⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 16	●●⋆●⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆●⋆⋆	●●⋆●	⋆⋆⋆○○○⋆○	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 18	●●●○⋆●⋆⋆●⋆⋆○⋆⋆⋆⋆⋆○	●⋆●⋆	⋆○○⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	○○⋆●	●●●●●●●	⋆⋆⋆⋆
1111111 2 20	●●⋆●●⋆⋆○⋆●⋆⋆⋆⋆⋆⋆⋆⋆	●●⋆⋆	○⋆⋆⋆○⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 22	●●⋆○⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆	●⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 24	●●●●⋆●⋆●⋆⋆⋆●⋆⋆⋆○⋆⋆	●●●○	⋆⋆⋆○○○○⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	●●⋆●	●●●●●●●	●⋆○○
1111111 2 26	●●⋆○⋆⋆⋆⋆⋆⋆⋆⋆●⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 28	●●⋆●⋆⋆●○⋆⋆⋆⋆⋆●⋆⋆⋆⋆	●○⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 30	●●●○●●⋆⋆⋆●⋆○⋆⋆○⋆⋆⋆	●⋆○⋆	○⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	○○⋆●	●●●●●●●	⋆⋆⋆⋆
1111111 2 32	●●⋆●⋆⋆⋆●⋆⋆⋆⋆⋆⋆⋆○⋆⋆	●●⋆○	⋆⋆⋆○○○⋆○	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 34	●●⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆○⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 36	●●●●⋆●⋆○○⋆⋆○⋆⋆⋆⋆⋆○	●●○⋆	⋆○○⋆○⋆○⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	●●⋆●	●●●●●●●	●⋆○○
1111111 2 38	○○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆○⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 40	●●⋆●●⋆⋆○⋆○⋆⋆⋆⋆⋆○⋆⋆	●●⋆○	○⋆⋆○○○⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 42	●●○○⋆○○⋆⋆⋆⋆○⋆○⋆⋆⋆⋆	●⋆○⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	○○⋆●	●●●●●●●	⋆⋆⋆⋆
1111111 2 44	●●⋆○⋆⋆⋆○⋆⋆○⋆⋆⋆⋆⋆⋆⋆	○○⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●●⋆⋆	●●●●●●●	●⋆○○
1111111 2 46	○○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆●○	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 48	●●○●⋆●⋆○⋆⋆⋆○⋆⋆⋆○⋆⋆	●○○○	⋆⋆⋆○○○○○	⋆●●	●●●●●●●●●●●●	⋆●⋆●⋆	●●⋆●	●●●●●●●	●⋆○○
1111111 2 50	●●⋆○○⋆⋆⋆⋆○⋆⋆⋆⋆⋆⋆⋆⋆	●⋆⋆⋆	○⋆⋆⋆⋆⋆⋆⋆	⋆●○	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	○○⋆⋆	●●●●●●●	⋆⋆⋆⋆
1111111 2 52	○●⋆○⋆⋆⋆○⋆⋆⋆⋆○⋆⋆⋆⋆⋆	○○⋆⋆	⋆⋆⋆⋆○⋆⋆⋆	⋆●●	●●●●●●●●●●●●	⋆⋆⋆⋆⋆	●○⋆⋆	●●●●●●●	●⋆○○
1111111 2 54	●●●○⋆○⋆⋆○⋆⋆○⋆⋆⋆⋆⋆○	○⋆○⋆	⋆○○⋆⋆⋆⋆⋆	⋆○⋆○⋆●⋆○⋆●⋆●⋆○⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆●⋆○○	●●●●●●●●●○●●	⋆○⋆○⋆	○○⋆●	●●●●●●●	⋆⋆⋆⋆
1111111 2 56	○○⋆○⋆⋆○○⋆⋆⋆⋆⋆○⋆○⋆⋆	○○⋆○	⋆⋆⋆○○○⋆⋆	⋆●⋆○⋆○⋆●⋆●⋆○⋆○⋆●⋆●⋆●⋆○⋆●⋆●⋆○⋆●⋆●⋆○⋆●⋆○⋆●○	○●●●●●○●●●●●	⋆⋆⋆⋆⋆	○●⋆⋆	●●○○●●○	●⋆○○
1111111 2 58	○○⋆○⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆	○⋆⋆⋆	⋆⋆⋆⋆⋆⋆⋆⋆	⋆○⋆○⋆○⋆○⋆○⋆○⋆○⋆○⋆○⋆○⋆●⋆○⋆●⋆○⋆○⋆●⋆○⋆●⋆○⋆○○	●●●○●○●○●○○○	⋆⋆⋆⋆⋆	○○⋆⋆	○○●●○●●	⋆⋆⋆⋆

Table A.6
Continued.

s / s'	000000000000000000 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 111111111111111111 000000000111111111 123456789012345678	0000 1111 1111 1111 1111 1111 2222 0000 2468	00000000 11111111 11111111 11111111 11111112 11112222 23342222 10000000 03642464	11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 11111111111111111111111111111111111111111 00000000011111111112222222222333333333344 12345678901234567890123456789012345678902	111111111111 111111111111 111111111111 111111111111 111111111111 111111111111 222222222222 000011111222 246802468024	11111 11111 11111 11111 11111 11111 33333 00011 36925	1111 1111 1111 1111 1111 1111 4456 0000 4856	1111111 1111111 1111111 1111111 1111111 2222222 2222222 0000111 2468024	1111 1111 1111 1111 1122 2322 4322 0000 4324
1111111 3 3	●★●★★★★★○★★★★★★★★★	★★★★	★●★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 6	●●●★★●★★○★★★★★★★★○	★★★★	★●●★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●●●●	★★★●	★★★★★★★	★●★★
1111111 3 9	●★●★★★★★●★★★★★★★★★	★★★★	★●★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 12	●●●●★●★★○★★●★★★★★○	○★○★	★●○★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●●●●	★★★●	★★○★★○★	★●★★
1111111 3 15	●★●★●★★★○★★★★★★○★★	★★★★	★●★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 18	●●●★★●★★●★★★★★★★★○	★★★★	★●○★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●●●●	★★★●	★★★★★★★	★●★★
1111111 3 21	●★●★★★●★○★★★★★★★★★	★★★★	★○★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 24	●●●●★●★●○★★○★★★★★○	○○○★	★○○★○★○★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●●●●	★★★●	★★○★★○★	★●★★
1111111 3 27	●★●★★★★★●★★★★★★★★★	★★★★	★●★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 30	●●●★○●★★○○★★★★○★★○	★★★★	★○○★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●●●●	★★★●	★★★★★★★	★●★★
1111111 3 33	○★○★★★★★○★○★★★★★★★	★★★★	★○★★★★★★	★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★★○★★●★★●★●	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111111 3 36	●●●○★○★★○★★○★★★★★○	○★○★	★○○★★★★★	★★●★★●★★○★★●★★●★★●★★●★★●★★●★★●★★●★★●★★●★●	★★●★★●★★●★★●	●●○●○	★★★●	★★○★★○★	★●★★
1111111 3 39	○★○★★★★★○★★★○★★★★★	★★★★	★○★★★★★★	★★●★★○★★○★★○★★●★★○★★○★★●★★○★★○★★○★★●★★○★●	★★★★★★★★★★★★	●●●○●	★★★★	★★★★★★★	★○★★
1111111 4 4	●●★●★★★○★★★★★★★○★★	●●★○	★★★●○○★★	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★★★	●●★★	○●○●○●○	●★○○
1111111 4 8	●●★●★★★●★★★★★★★○★★	●●★●	★★★●○○★○	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★★★	●●★★	○●○●○●○	●★○○
1111111 4 12	●●●●★●★○★★★●★★★○★★	●●●○	★★★○○○○★	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★●★	●●★○	○●○●○●○	●★○○
1111111 4 16	●●★●★★★●★★★★★★★○★★	●●★○	★★★○○○★○	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★★★	●●★★	○●○●○●○	●★○○
1111111 4 20	●●★●●★★○★○★★★★★○★★	●●★○	○★★○○○★★	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★★★	●●★★	○●○●○●○	●★○○
1111111 4 24	●●○●★●★●★★★○★★★○★★	●○○○	★★★○○○○○	★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★★★●★	○●○●○●○●○●○●	★★★●★	●●★○	○●○●○●○	●★○○
1111111 4 28	○○★○★★○○★★★★★○★○★★	○○★○	★★★○○○★★	★★★○★★★●★★★○★★★○★★★●★★★○★★★○★★★○★★★●★★★○★	○○○○○●○●○○○○	★★★★★	●●★★	○●○○○○○	○★○○
1111111 5 5	●★★★●★★★★★★★★★★★★★	★★★★	★★★★★★★★	★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★	★★★★★★★★★★★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 5 10	●●★★●★★★★○★★★★★★★★	★★★★	★★★★★★★★	★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★	★★★★●★★★★●★★	★★★★★	★★●★	★★★★★★★	★★★★
1111111 5 15	●★●★●★★★★★★★★★★○★★	★★★★	★★★★★★★★	★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★★★★●★	★★★★★★★★★★★★	★★★★●	★★●★	★★★★★★★	★★★★
1111111 5 25	○★★★○★★★★★★★★★★★★★	★★★★	★★★★★★★★	★★★★●★★★★○★★★★●★★★★●★★★★○★★★★●★★★★●★★★★●★	★★★★★★★★★★★★	★★★★★	★★○★	★★★★★★★	★★★★
1111111 6 6	●●●○★●★★○★★○★★★★★○	●★●★	★●○★★★★★	★★★★★●★★★★★●★★★★★●★★★★★●★★★★★●★★★★★●★★★★●	★★●★★●★★●★★●	★●★●★	★★★●	★★●★★●★	★★★★
1111111 6 12	●●●●★●★○○★★○★★★★★○	●●○★	★○○★○★○★	★★★★★●★★★★★●★★★★★●★★★★★●★★★★★●★★★★★●★★★★●	★★●★★●★★●★★●	★●★●★	★★★●	★★●★★●★	★★★★
1111111 6 18	○●●○★○★★○★★○★★★★★○	○★○★	★○○★★★★★	★★★★★○★★★★★●★★★★★●★★★★★●★★★★★●★★★★★●★★★★●	★★●★★●★★●★★○	★●★○★	★★★●	★★●★★●★	★★★★
1111111 7 7	●★★★★★○★★★★★★★★★★★	★★★★	★★★★★★★★	★★★★★★●★★★★★★●★★★★★★●★★★★★★●★★★★★★●★★★★★●	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 7 14	●●★★★★○★★★★★★○★★★★	★★★★	★★★★★★★★	★★★★★★●★★★★★★●★★★★★★●★★★★★★●★★★★★★●★★★★★●	★★★★★★●★★★★★	★★★★★	★★★★	★★★★★★★	★★★★
1111111 8 8	●●★●★★★●★★★★★★★○★★	●●★○	★★★○○○★○	★★★★★★★●★★★★★★★●★★★★★★★●★★★★★★★●★★★★★★★●★	★○★●★○★●★○★●	★★★★★	○●★★	○○○●○○○	○★○○
1111111 8 16	○○★○★★★●★★★★★★★○★★	○○★○	★★★○○○★○	★★★★★★★●★★★★★★★○★★★★★★★●★★★★★★★○★★★★★★★●★	★○★○★○★○★○★○	★★★★★	○●★★	○○○○○○○	○★○○
1111111 9 9	●★●★★★★★●★★★★★★★★★	★★★★	★●★★★★★★	★★★★★★★★●★★★★★★★★●★★★★★★★★●★★★★★★★★●★★★★★	★★★★★★★★★★★★	○○●○○	★★★★	★★★★★★★	★○★★
1111111 10 10	●●★○○★★★★○★★★★★★★★	●★★★	○★★★★★★★	★★★★★★★★★●★★★★★★★★★●★★★★★★★★★●★★★★★★★★★●★	★★★★●★★★★●★★	★★★★★	★★★★	★★★★●★★	★★★★
1111112 2 2	★●★○★★★○★★★★★★★★★★	●○★★	★★★★●★★★	★★★	●●●●●●●●●●●●	★★★★★	○○★★	●●●●●●●	○★●●
1111112 2 4	★●★●★★★○★★★★★★★○★★	●●★○	★★★○●●★★	★★★	●●●●●●●●●●●●	★★★★★	●●★★	●●●●●●●	●★●●
1111112 2 6	★●★○★●★○★★★○★★★★★★	●○●★	★★★★●★●★	★★★	●●●●●●●●●●●●	★★★★★	○○★●	●●●●●●●	○★●●
1111112 2 8	★●★●★★★●★★★★★★★○★★	●●★●	★★★○●●★○	★★★	●●●●●●●●●●●●	★★★★★	●●★★	●●●●●●●	●★●●
1111112 2 10	★●★○★★★○★●★★★★★★★★	●○★★	○★★★●★★★	★★★	●●●●●●●●●●●●	★★★★★	○○★★	●●●●●●●	○★●●
1111112 2 12	★●★●★●★○★★★●★★★○★★	●●●○	★★★○●●○★	★★★	●●●●●●●●●●●●	★★★★★	●●★●	●●●●●●●	●★●●
1111112 2 14	★●★○★★★○★★★★★●★★★★	●○★★	★★★★●★★★	★★★	●●●●●●●●●●●●	★★★★★	○○★★	●●●●●●●	○★●●
1111112 2 16	★●★●★★★●★★★★★★★○★★	●●★○	★★★○●○★○	★★★	●●●●●●●●●●●●	★★★★★	●●★★	●●●●●●●	●★●●
1111112 2 18	★●★○★●★○★★★○★★★★★○	●○○★	★★○★●★○★	★★★	●●●●●●●●●●●●	★★★★★	○○★●	●●●●●●●	○★●●
1111112 2 20	★●★●★★★○★○★★★★★○★★	●●★○	○★★○●○★★	★★★	●●●●●●●●●●●●	★★★★★	●●★★	●●●●●●●	●★●●
1111112 2 22	★○★○★★★○★★★★★★★★★★	○○★★	★★★★○★★★	★★★	●●●●●●●●●●○●	★★★★★	○○★★	●●●●●●●	○★●●
1111112 2 24	★●★●★●★○★★★○★★★○★★	●○○○	★★★○○○○○	★★★	●●●●●●●●●●●●	★★★★★	●●★●	●●●●●●●	●★●●
1111112 2 26	★○★○★★★○★★★★★★★★★★	○○★★	★★★★○★★★	★★★	●●●●●●●●●○●○	★★★★★	○○★★	●●●●●●●	○★●●
1111112 2 30	★○★○★○★○★○★○★★★★★★	○○○★	○★★★○★○★	★★★	●○●●○●●○●○●○	★★★★★	○○★●	○●●●○●○	○★●○
1111112 4 4	★●★●★★★○★★★★★★★○★★	●●★○	★★★●●●★○	★★★	★●★●★●★●★●★●	★★★★★	●●★★	○●○●○●○	●★○●
1111112 4 8	★●★●★★★●★★★★★★★○★★	●●★○	★★★○●○★○	★★★	★●★●★●★●★●★●	★★★★★	●●★★	○●○●○●○	●★○●
1111112 4 12	★●★●★●★○★★★○★★★○★★	●○○○	★★★○○○○○	★★★	★●★●★●★●★●★○	★★★★★	●●★★	○●○●○●○	○★○●
1111112 6 6	★●★○★●★○★★★○★★★★★○	●○○★	★★○★●★○★	★★★	★★●★★●★★●★★●	★★★★★	★★★●	★★●★★●★	★★★★
1111112 8 8	★○★○★★★●★★★★★★★○★★	○○★○	★★★○○○★○	★★★	★★★●★★★●★★★●	★★★★★	○●★★	★○★○★○★	○★○○
1111113 3 3	★★●★★★★★○★★★★★★★★★	★★★★	★●★★★★★★	★★★	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111113 3 6	★★●★★●★★○★★★★★★★★○	★★★★	★●○★★★★★	★★★	★★★★★★★★★★★★	●●●●●	★★★●	★★★★★★★	★●★★
1111113 3 9	★★●★★★★★●★★★★★★★★★	★★★★	★●★★★★★★	★★★	★★★★★★★★★★★★	●●●●●	★★★★	★★★★★★★	★●★★
1111113 3 12	★★●★★○★★○★★○★★★★★○	★★○★	★○○★★★★★	★★★	★★★★★★★★★★★★	●●●○●	★★★●	★★★★★★★	★○★★
1111114 4 4	★★★●★★★○★★★★★★★○★★	○●★○	★★★○○○★○	★★★	★★★★★★★★★★★★	★★★★★	●●★★	★○★○★○★	●★○○
1111114 4 8	★★★○★★★●★★★★★★★○★★	○○★○	★★★○○○★○	★★★	★★★★★★★★★★★★	★★★★★	●●★★	★○★○★○★	○★○○
1111122 2 2	★★★★★★★★★★★★★★★★★★	●○★○	★★★○●○★★	★★★	★★★★★★★★★★★★	★★★★★	★★★★	●●●●●●●	○★●●
1111122 2 4	★★★★★★★★★★★★★★★★★★	●●★○	★★★○●●★●	★★★	★★★★★★★★★★★★	★★★★★	★★★★	●●●●●●●	●★●●
1111122 2 6	★★★★★★★★★★★★★★★★★★	●○●○	★★★○●○○★	★★★	★★★★★★★★★★★★	★★★★★	★★★★	●●●●●●●	○★●●
1111122 2 8	★★★★★★★★★★★★★★★★★★	●●★○	★★★○●○★○	★★★	★★★★★★★★★★★★	★★★★★	★★★★	●●●●●●●	●★●●
1111122 2 10	★★★★★★★★★★★★★★★★★★	●○★○	○★★○●○★★	★★★	★★★★★★★★★★★★	★★★★★	★★★★	●●●●●●●	○★●●
1111122 2 12	★★★★★★★★★★★★★★★★★★	●○○○	★★★○○○○○	★★★	★★★★★★★★★★★★	★★★★★	★★★★	○●●●○●○	○★●●
1111122 4 4	★★★★★★★★★★★★★★★★★★	●●★○	★★★○●○★○	★★★	★★★★★★★★★★★★	★★★★★	★★★★	★●★●★●★	●★○●
1111122 6 6	★★★★★★★★★★★★★★★★★★	○○○○	★★★○○○○★	★★★	★★★★★★★★★★★★	★★★★★	★★★★	★★●★★●★	★★★★
1111133 3 3	★★★★★★★★★★★★★★★★★★	★★★★	★●★★★★★★	★★★	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★●★★
1111222 2 2	★★★★★★★★★★★★★★★★★★	★★★★	★★★★●○★○	★★★	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★●●
1111222 2 4	★★★★★★★★★★★★★★★★★★	★★★★	★★★★●○★○	★★★	★★★★★★★★★★★★	★★★★★	★★★★	★★★★★★★	★★●●

Table A.7
Lower bounds for a_n and matrices with high extension spectra, $10 \leq n \leq 19$.

n	$a_n \geq$	$\|\mathcal{L}_{n-1}\|$	$\det A_{n-1}$	$a_{A_{n-1}}$	A_{n-1}
10	259	2	110	257	$[7, 39, 5A, 9C, E1, 149, 174, 193, 1AA]$
11	739	6	291	679	$[F, 71, B6, 13A, 1C3, 1DC, 256, 299, 2EC, 325]$
12	2107	19	779	1894	$[F, 73, 195, 1EA, 2A6, 35C, 4D6, 53E, 565,$ $6B9, 703]$
13	6157	18	2201	5618	$[1F, E3, 17C, 3A5, 649, 6D6, 732, A6E, AB8,$ $B53, C35, D8E]$
14	19073	40	6731	16821	$[3F, 1C7, 2D9, 76A, C4D, CF2, F94, 1575,$ $168E, 195A, 19A9, 1A64, 1E13]$
15	58741	46	23288	53117	$[7D, 38F, 5B2, ED5, 189B, 19E4, 1F29, 2AEA,$ $2D1C, 32B4, 3353, 34C9, 3C27, 164E]$
16	185693	190	67832	161599	$[FD, 71F, BE3, 1D29, 324F, 36B2, 3995, 5370,$ $55C6, 5A9A, 61AB, 6C53, 6E24, 27C8, 297E]$
17	610187	480	213175	517794	$[1FB, E3E, 17C6, 3A53, 649F, 6D64, 732B,$ $A6E2, AB8D, B535, C356, D8A7, DC49, 4F91,$ $72FC, F99A]$
18	2039033	697	709503	1719277	$[3F9, 1C7E, 2D95, 76AC, C4D2, CF27, F949,$ $15755, 168E5, 195A3, 19A9C, 1A64B, 1E13E,$ $14D8A, 17A33, 33C6, 1AF70]$
19	6478579	54	2331887	4663774	$[7E9, 38F7, 5F13, E95D, 19277, 1B599, 1CCAF,$ $29B8E, 2AE31, 2D4C5, 30D56, 3629F, 37125,$ $13E4C, 1EBC2, 358F8, 2F46A, E7B4]$